A Brief Theory of Limit Cycles

A DISSERTATION SUBMITTED
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE
BY

ANINDYAMOY MAITY

ROLL NO - 111PG2221541802
REGISTRATION NO: 1071711400084 OF 2017-2020

UNDER THE SUPERVISION
OF
DR. SHYAMAL SEN

DEPARTMENT OF MATHEMATICS
BRAHMANANDA KESHAB CHANDRA COLLEGE
KOLKATA-700108, WEST BENGAL
(AFFILIATED TO WEST BENGAL STATE UNIVERSITY)

2024

Brahmananda Keshab Chandra College
Kolkata

Certificate

I, Anindyamoy Maity, a student of M. Sc in Mathematics (4th Semester, January to June 2024) registered (Roll No. 111PG2221541802 Registration No. 1071711400084 of Year 2017 – 2020) at Brahmananda Keshab Chandra College, Kolkata under the West Bengal State University, pursued my dissertation work in the form of review writing under the guidance of Dr.Shyamal Sen. I, hereby, submit the review entitled 'A Brief Theory of Limit Cycles' as per the curriculum of the University for partial fulfillment of the degree Master of Science in Mathematics.

Anindyamoy Maity
Anindyamoy Maity

Place: Kolkata - 108
Date : 09/08/2024

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Shyamal Sen 09.08.2024
(Dr. Shyamal Sen)
Supervisor
Associate Professor,
Department of Mathematics,
Brahmananda Keshab Chandra College,
Kolkata- 700108

Examined
09/08/2024

09/08/2024
Principal
Brahmananda Keshab Chandra College
Kolkata - 700108

Acknowledgment

I acknowledge my deepest regards, sincere appreciation and heartfelt thanks to my supervisor Dr. Shyamal Sen for his kind supervision and guidance. Without his active guidance it would never be possible for me to complete the work.

I would like to express my sincere thanks and gratitude to Dr. Saral Datta, Head of the Department of Mathematics, Brahmananda Keshab Chandra College, for providing me with requisite facilities to carry out my research work. I would like to acknowledge my gratitude to Dr.Neeta Pandey specially and the faculty members of the Department for their valuable suggestions and supports.

Last but not the least I would also like to thank my friends and my fellow classmates who helped me a lot during the period of preparing the project.

Date:05.08.2024 Anindyamoy Maity

. Roll No. - 111PG2221541802

. Registration No.-1071711400084

Contents

Chapter 1

Flow Of the Vector Field

1.1 Introduction

In this chapter some basic fundamental definitions will be described which will be repeatedly required to explain the next chapters . The main concentrated topic is the limit cycle . The limit cycle is a special phenomenon of a dynamical system (at least in two dimensions) .
Another thing should be remembered that non-linear dynamics needs pure mathematical approaches . Some famous authors have already shown the unfathomable abstractness in their books . Few books related to the non-linear dynamics will provide a introductory discussion which are undoubtedly helpful as beginner's guide but unfortunately they tend to avoid the topological and other categorical discussions which is more precise . One should also realise that those definitions are of smaller mathematical classes . Here in this chapter we have tried to define everything with simple attempts but those abstractness could not be removed because its richness is hidden in those archaic definitions only . And readers are expected to explore and accept them without any hesitations . We are definitely not exaggerating the contents here rather we will focus upon only a few things we need further repeatedly.

1.2 Smooth functions

A real-valued funtion $f : U \to \mathbb{R}$ (U is an open set in $\mathbb{R}^n$) is said to be C^K at $p \in U$ if its partial derivatives $\dfrac{\partial^j f}{\partial x_{i_1} \cdot \partial x_{i_2} \ldots \partial x_{i_j}}$ $\forall$ j $\leq k$ (also satisfying $i_1 + i_2 + \cdots + i_n = j$) exist and are continuous .
If the condition holds $\forall\ k \geq 0$, then the function $f : U \to \mathbb{R}$ is said to be C^∞ or commonly known as *smooth* function as well . The smoothness can be defined globally or locally as needed . Suppose $X \subseteq \mathbb{R}^K$ and $Y \subseteq \mathbb{R}^L$ be any two topological spaces . Let U be an open neighbourhood in $\mathbb{R}^K$ and define a smooth map $g : U \to \mathbb{R}^L$ so that $f|_{U\cap X} = g|_{U\cap X}$

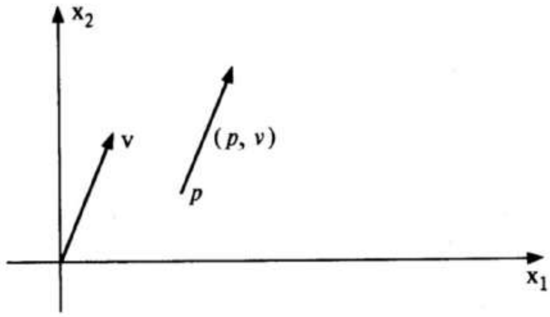

Figure 1.1: Vector at a point p

1.3 Smooth vector fields

A *vector* at a point $p \in \mathbb{R}^n$ is denoted by the pair $\mathbf{v} = (p, v)$ where $v \in \mathbb{R}^n$. Geometrically consider a $\mathbf{v}$ as the vector v translated so that its tail is at p rather than at the origin . Then a smooth vector field $\mathbf{X}$ on an open set $U \subset \mathbb{R}^n$ is a function which assigns each point of U to a vector at that point . The vector field can also be defined by the smooth map as $\mathbf{X} : U \to \mathbb{R}^n$ by $\mathbf{X}(p) = (p, X_p)$

1.4 Velocity vector of a parameterized curve and a integral curve

A *parametrized curve* in an open set $U \in \mathbb{R}^n$ is a smooth function $\Psi : (-\epsilon, \epsilon) \to U$ so that $\Psi(0) = s$ and ϵ could be ∞ and defined by $\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_n(t))$

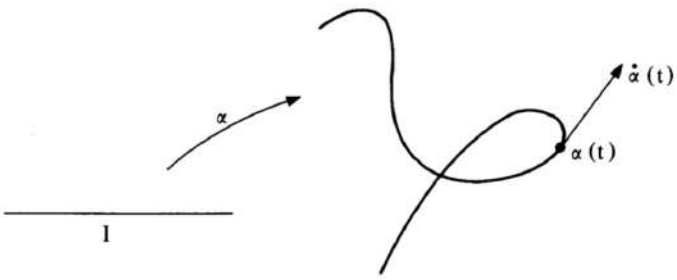

Figure 1.2: Velocity vector of a parametrized curve in $\mathbb{R}^2$

Let $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$. Then the Jacobian of the smooth map $d\Psi|_0$: $\mathbb{R} \to \mathbb{R}^n$ is obtained by

$$\begin{pmatrix} \frac{d\psi_1}{dt}|_0 \\ \frac{d\psi_2}{dt}|_0 \\ \cdots \\ \frac{d\psi_n}{dt}|_0 \end{pmatrix}$$

This implies that

$$d\Psi|_0(1) = (\frac{d\psi_1}{dt}|_0, \frac{d\psi_2}{dt}|_0, \ldots, \frac{d\psi_n}{dt}|_0) = \dot{\Psi}(0)$$

Then $d\Psi|_0(1) = (\frac{d\psi_1}{dt}|_0, \ldots, \frac{d\psi_n}{dt}|_0)$ is the velocity vector at $t = 0$ and also note that $\dot{\Psi}(0) = s \in T_sU$ where T_sU represents the *tangent space* of U at s.

Also an *integral curve* is a parametrized curve $\Psi : I \to \mathbb{R}^n$ of the smooth vector field $\mathbf{X} : U \to \mathbb{R}^n$ defined by $\dot{\Psi}(t) = \mathbf{X}(\Psi(t))$, where $I \subset \mathbb{R}$

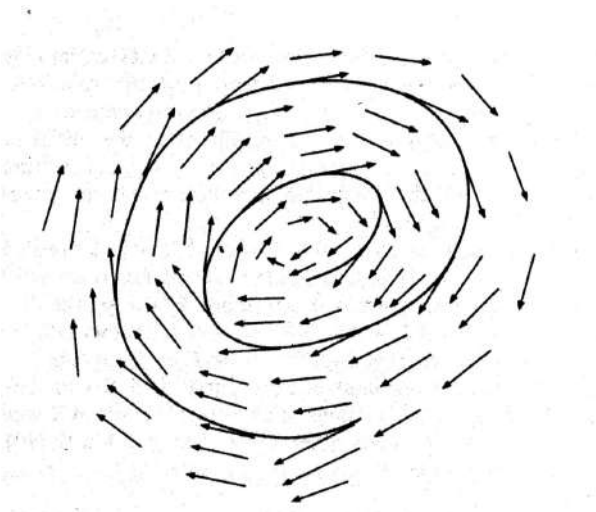

Figure 1.3: An integral curve of a vector field

1.4.1 Example

Let $\mathbf{X}$ be a smooth vector field

$$\mathbf{X}(x_1, x_2) = (x_1, x_2)$$

defined on $\mathbb{R}^2$. Let $p = (a, b) \in \mathbb{R}^2$. Find the maximal integral curve[1] .

According to the previous section the differential equations associated with the vector field can be wriiten as

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \\ \frac{dx_2}{dt} &= x_2 \end{aligned} \tag{1.1}$$

with the initiatial conditions $x_1(0) = a$ and $x_2(0) = b$.
Hence the general solution of (1.1) is given by

$$\begin{aligned} x_1 &= c_1 e^t \\ x_2 &= c_2 e^t \end{aligned} \tag{1.2}$$

$\forall\ t \in \mathbb{R}$ and c_1 and c_2 are the arbitrary constants . Using initial conditions we obtain $c_1 = a$ and $c_2 = b$. Hence the maximal integral curve of $\mathbf{X}$ through $p(0, 0)$ is a smooth map $\Psi : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\Psi(t) = (ae^t, be^t)\ \forall\ t \in \mathbb{R}$$

Then $\mathbf{X}$ is called a complete vector field.[2]

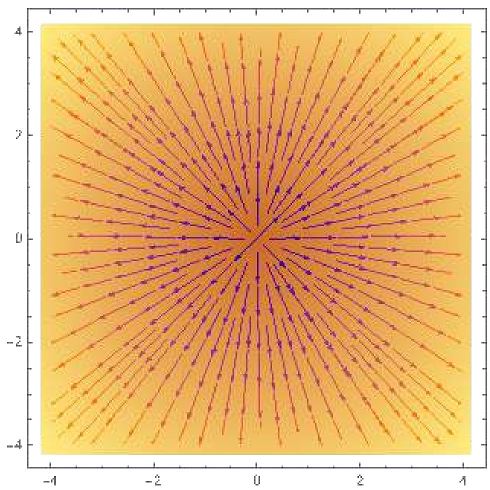

Figure 1.4: Graphical plot of a complete vector field

[1]An integral curve $\alpha : I \to \mathbb{R}^n$ of $\mathbf{X}$ through $p \in \mathbb{R}$ if there exists an integral curve $\beta : J \to R^n$ through p so that $J \subset I$ and $\alpha|_J = \beta$

[2]A smooth vector field is said to be complete if there exists a maximal integral curve having its domain as $\mathbb{R}$

1.5 Picard-Lindeloff theorem

let $\mathbf{X}$ be a smooth vector field completely determined by the corresponding smooth map $\mathbf{X} : U \to \mathbb{R}^n$. An integral curve of $\mathbf{X}$ passing through a point $p \in \mathbb{R}^n$ is a smooth function $\Psi : I \to U$ of the differential equation

$$\dot{\Psi} = \mathbf{X}(\Psi(t)) \tag{1.3}$$

$\forall\, t \in I \subset \mathbb{R}$ with initial condition $\Psi(0) = p$.
Let $V \subset \mathbb{R}^n$ be open so that $p \in V \subset \overline{V} \subset U$. Then Picard's theorem for ODE(ordinary differential equations) says that there exists a unique smooth map

$$\Phi : (-\epsilon_p, \epsilon_p) \times \overline{V} \to U$$

for some $\epsilon_p > 0$ such that for each $q \in \overline{V}$,
$\gamma_q : (-\epsilon_p, \epsilon_p) \to U$ defined $\gamma_q(t) = \Phi(t, q)\ \forall\, t \in (-\epsilon_p, \epsilon_p)$ satisfying

$$\dot{\gamma}_q(t) = \mathbf{X}(\gamma_q(t)) \tag{1.4}$$

with initial condition $\gamma_q(0) = q$.
Evidently , γ_q integral curve of $\mathbf{X}$ through q , $\forall\, q \in \overline{V}$.

Chapter 2

Limit Sets and Attractors

2.1 Introduction

In this chapter we will undergo through one of most significant topics that deals with limit sets , attracting sets and attractors etc . These few topics are the parts and parcels in the whole chapter and it will help us to develop the upcoming theories and to enter the relatively deep part of the theory of limit cycles .

2.2 Periodic orbits

Let U be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is a smooth map . Consider the smooth vector field ,

$$\dot{x} = f(x) \tag{2.1}$$

$\forall \; x \;\; \in \mathbb{R}^n$ and $f \in C^\infty(U)$, where $C^\infty(U)$ is a set of continuously differentiable functions on a set U
Now consider the smallest value of time $T(> 0)$ so that the solution curve passing through an initial point x_0 can be written as $\phi(t, x_0)$ where the flow $\phi(t, x_0)$ satisfies the relation

$$\phi(t + T, x_0) = \phi(t, x_0) \tag{2.2}$$

then the curve is called a *'cycle'* or an *'orbit'*, precisely a periodic orbit of period T . The solution curve is denoted by Γ.

2.3 Limit sets

Limit sets play an important role in the study of non-linear dynamical system and to be precised it is a part and parcel and foundational aspect of the theory of limit cycles .
Consider the autonomous system as given in (2.1) . Let $\phi(* \;, x)$ defines a dynamical system[1] on an open set U of $\mathbb{R}^n$ such that $\phi(* \;, x) : \mathbb{R} \to U$ is a function that defines

[1]The point can represent several points those which move as t varies

a solution curve of (2.1). Now a trajectory Γ passing through $x_0 \in U \subset \mathbb{R}^n$ can be presented as

$$\Gamma_{x_0} = \{x \in U : x = \phi(t, x_0), t \in \mathbb{R}\} \tag{2.3}$$

We shall refer Γ_{x_0} as the trajectory of (2.1) passing through x_0 at time $t = 0$.
From the above definition , two other definitions can be extracted as :

2.3.1 Positive half-trajectory

By the positive half-trajectory through an initial condition $x_0 \in U$ we exactly mean the motion along the curve , when t is non-negative

$$\Gamma_{x0}^{+} = \{x \in U : x = \phi(t, x_0), t \geq 0\} \tag{2.4}$$

2.3.2 Negative half-trajectory

By the negative half-trajectory through an initial condition $x_0 \in U$ we exactly mean the motion along the curve , when is t is non-positive

$$\Gamma_{x0}^{-} = \{x \in U : x = \phi(t, x_0), t \leq 0\} \tag{2.5}$$

The idea of limit sets is one of the most important concepts for the definition of a limit cycle . There are two types of limit sets , namely ω-limit set and α-limit set . These types of limit sets are defined in the following sections.

2.3.3 ω-limit sets

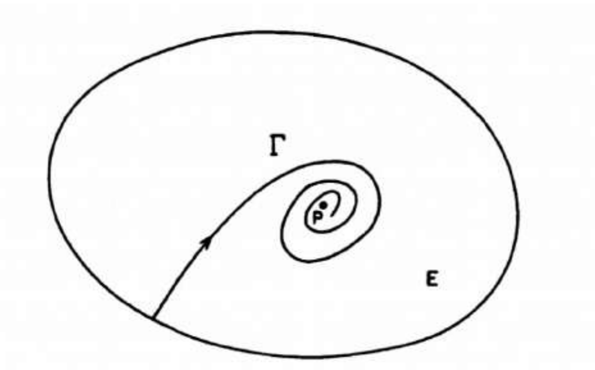

Figure 2.1: The trajectory approaches the point p positively

Any arbitrary point p in U is said to be an ω-limit point of the trajectory $\phi(* \, , x)$ of the system (2.1) if there exists a sequence $\{t_n\}_{n=1}^{\infty} \uparrow \infty$ (tending as increasingly or isotonically) such that

$$\lim_{n\to\infty} \phi(t_n, x) = p \tag{2.6}$$

Now construct a set of all such ω-limit points of a trajectory Γ . This set is called ω-limit set of the trajectory Γ, denoted as $\omega(\Gamma)$. Similarly an α-limit set can be defined as follows :

2.3.4 α-limit set

Any arbitrary point q in U is said to be an α-limit point of the trajectory $\phi(* \, , x)$ of the system (2.1) if there exists a sequence $\{t_n\}_{n=1}^{\infty} \downarrow$-$\infty$ (tending decreasingly or antitonically) such that

$$\lim_{n\to-\infty} \phi(t_n, x) = q \tag{2.7}$$

Similarly the set of all such α-limit points of a trajectory Γ can be constructed . This set is called α-limit set of the trajectory Γ , denoted as $\alpha(\Gamma)$.

Thus we have two kinds of limit sets now. So obviously we obtain a set of all kinds of limit points together and denote it by $L_{\alpha}^{\omega}(\Gamma)$ or just L_{α}^{ω} where

$$L_{\alpha}^{\omega}(\Gamma) = \omega(\Gamma) \cup \alpha(\Gamma) \tag{2.8}$$

L_{α}^{ω} is called a ***limit set*** for a trajectory of a smooth system .

2.4 General topological properties of limit sets

2.4.1 Theorem

The α and ω-limit sets of a trajectory of the system as given in (2.1) are closed in $\mathbb{R}^n$.
Proof

Let $p \in \overline{\omega(\Gamma)}$. Then there exists an open ball of radius $\epsilon(>0)$ centered at p denoted by $B_{\epsilon}(p)$. And clearly , $B_{\epsilon}(p) \cap \omega(\Gamma) \neq \phi$. We will show that $p \in \omega(\Gamma)$.Then we can find a sequence $\{p_n\}_{n=1}^{\infty}$ in $\omega(\Gamma)$ such that $||p_n - p|| \to 0$, as $n \to \infty$. Let x_n denote the points on Γ.
But since $\{p_n\}_{n=1}^{\infty}$ is in $\omega(\Gamma)$ there exists a sequence $\{t_k^{(n)}\}_{k=1}^{\infty}$ such that $\lim\limits_{n\to\infty} \phi(t_k^{(n)}, x_n) = p_n$ that for any $\eta > 0$, there exists natural number $k_0(\eta)$ depending upon η,say k_0 and satisfies the relation $||\phi(t_k^n, x_n) - p_n|| < \eta$, for $k \geq k_0(\eta)$. In particular choose $\eta = \frac{1}{n}$,then $k_0(\eta) = k_0(n)$ and satisfies the relation

$$||\phi(t_k^{(n)}, x_n) - p_n|| < \eta, \text{ for } k \geq k_0(n)$$

Now choose

$$k_1 = max\{k_0(1), k_0(2), \ldots, k_0(n)\}$$

Therefore , $||\phi(t_k^{(n)}, x_n) - p_n|| < \frac{1}{n}$, for $k \geq k_1$. Now compute

$$\begin{aligned} &||\phi(t_k^{(n)}, x_n) - p|| \\ &= ||\phi(t_k^{(n)}, x_n) - p_n + p_n - p|| \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Hence ,

$$\lim_{n\to\infty} \phi(t_k^{(n)}, x_n) = p \tag{2.9}$$

Thus by the definition of an ω-limit set $p \in \omega(\Gamma)$. So , $\omega(\Gamma)$ is closed .

Similar result can be shown for α-limit sets too .

2.4.2 Theorem

Let Γ be contained in a compact subset of $\mathbb{R}^n$. Then $\alpha(\Gamma)$ and $\omega(\Gamma)$ are connected and compact .

Proof

Let K be a compact subset of $\mathbb{R}^n$.Observe that due to the standard topology on $\mathbb{R}^n$ the compactness will be based on the generalised Heine-Borel theorem.
Now according the Statement $\Gamma \subset K$ $(K \subset \mathbb{R}^n)$. But $\lim_{t\to\infty} \phi(t_n, x_0) = p \in \omega(\Gamma)$, since $\omega(\Gamma)$ is closed . Hence $p \in K$ $[\because \phi(t_n, x_0) \in \Gamma \subset K]$

Thus , $\omega(\Gamma)$ is closed in K. But by weak hereditary property of compactness a closed subset of a compact set must be compact . So $\omega(\Gamma)$ is compact .
Similarly it can be shown that $\alpha(\Gamma)$ is compact .

We then observe that $\phi(*, x_0)$ is a smooth function on K and since K is compact , we say that ϕ is bounded . Now consider the sequence of points $\phi(t_n, x_0)$(through x_0) $\in$ K.

This is also a bounded sequence. Hence by Bolzano-Weirstrass theorem $\omega(\Gamma)$ is sequential compact . That is the sequence $\{\phi(t_n, x_0)\}_{n\in\mathbb{N}}$ has a convergent subsequence which converges in $\omega(\Gamma) \subset K$. Hence $\omega(\Gamma) \neq \phi$.
Now suppose $\omega(\Gamma)$ is disconnected and let A and B be two closed sets in $\omega(\Gamma)$ so that

$\omega(\Gamma) = A \cup B$. Let $p \in \omega(\Gamma)$, then p is an ω-limit point[2]. Again since $A \cap B = \phi$, this implies that either $p \in A$ or $p \in B$.
First assume that $p \notin A$, $i.e$ there exists at least one neighbourhood of p , say $B_{\delta_1}(p)$ so that $A \cap B_{\delta_1}(p) = \phi$ where $\delta_1 = \frac{\delta}{2}$ and $A \cap \overline{B_{\delta_1}(p)} \neq \phi$. But $p \in \omega(\Gamma)$. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} \phi(t_n, x_0) = p, x_0 \in \Gamma$.
Note that $B_{\delta_1}(p) = \{y : ||y-p|| < \frac{\delta}{2}\ \}$, for large n , $\phi(t_n, , x_0) \in B_{\delta_1}(p)$.
Now

$$\lim_{n\to\infty} d[\phi(t_n, x_0), A] = d[\lim_{n\to\infty} \phi(t_n, x_0), A] = d(p, A) = \frac{\delta}{2} = \delta_1 \tag{2.10}$$

Therefore

$$d(p, B) \geq d(A, B) - d(p, A) = \delta - \frac{\delta}{2} = \frac{\delta}{2} = \delta_1 \tag{2.11}$$

Hence , $B \cap B_{\delta_1}(p) = \phi \implies$ p $\notin B$ which contradicts the membership of p in $\omega(\Gamma)$.
Hence $\omega(\Gamma)$ must be connected.

2.4.3 Example

Suppose γ is a periodic orbit through p and let q be any point on it. Then for $t = t_k$ such that

$$\phi(t_k, p) = q \tag{2.12}$$

for some $t \in [0, T)$ where T is the period . And also

$$\phi(t_{k+1}, p) = q \tag{2.13}$$

if $t_{k+1} = t_k + T$
Hence every point of the orbit γ becomes a ω-limit point of γ .
So , q is a ω-limit point of γ .
$\therefore \omega(\gamma) = \gamma$ through p . To avoid the ambiguity it is better to write

$$\omega(\gamma(p)) = \gamma(p) \tag{2.14}$$

Similarly , it is also true for α-limit set i.e.

$$\alpha(\gamma(p)) = \gamma(p) \tag{2.15}$$

The above two properties (2.14) and (2.15) are called the ***Invariance of Limit Sets***.

[2] $\omega(\Gamma)$ is a closed and bounded subset of $\mathbb{R}^n$ by Heine-Borel theorem and again by Bolzano-Weirstrass theorem $\omega(\Gamma)$ has at least one limit point in $\mathbb{R}^n$

2.5 Invariant sets

Let X be a state space where all such dynamical phenomenons occur . A subset $S \subset X$ is then said to be an an invariant set of the system $\{T, X, \phi(t,*)\}$ if $x_0 \in S \implies \phi(t, x_0) \subseteq S$, $\forall\, t \geq 0$, where $x = x_0$ for $t = 0$. That is to say

$$\phi(t, S) \subseteq S \tag{2.16}$$

So an invariant set consists of the orbits of the dynamical system . Let Γ_{x_0} be an individual orbit passing through x_0 . We restrict the evolution operator say $\phi(t,*)$ to its invariant set S and call this operator as INV_E so that we have $INV_E : S \to S$ and denote the dynamical system as $\{T, S, INV_E\}$

2.5.1 Invariant torus

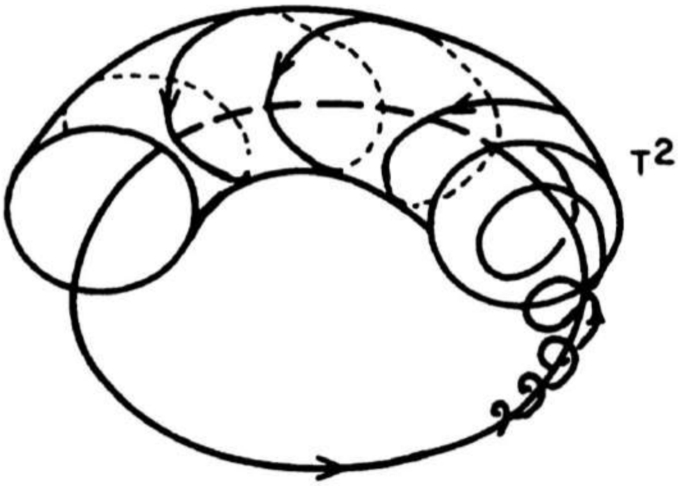

Figure 2.2: An invariant torus where the flow does not leave the manifold

One of the most graceful aspects of non-linear dynamics of the world is differential equations.Generally we do not even know how does a solution of a differential equation behave. It is the only prediction to some extent but largely there are certain equations whose behaviour after an initial period became so vague in nature with complex characteristics that most of the explanations are only with pen and papers and some computer programmings .
But they are not even predictable in classical sense . They break almost every rule and constitute some unexpected *'Chaotic behaviour'* those which are just describable .

2.6 Attracting set

A closed invariant set $A \subset U$ is said to be an attracting set of (2.1) if there exists some open neighborhood V of A such that $\forall\, x \in U$, $\phi(t, x) \in U$, $\forall\, t \geq 0$ and $\phi(t, x) \to A$ as

$t \to \infty$. That is equivalently it can also be stated as $d(\phi(t, x_0) \to 0$.
An attractor of (2.1) is an attracting set which contains a dense orbit. Observe that for any equilibrium point x_0 of (2.1) the attracting set is its own ω and α-limit sets since $\phi(t, x_0) = x_0, \forall\, t \in \mathbb{R}$.

2.6.1 Example

Consider the system

$$\begin{aligned} \dot{x} &= -y + x(1 - z^2 - x^2 - y^3) \\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2) \\ \dot{z} &= 0 \end{aligned} \tag{2.17}$$

To study the above system we make few slices and study the sliced planar system . We obtain the phase portrait in each sliced planar system

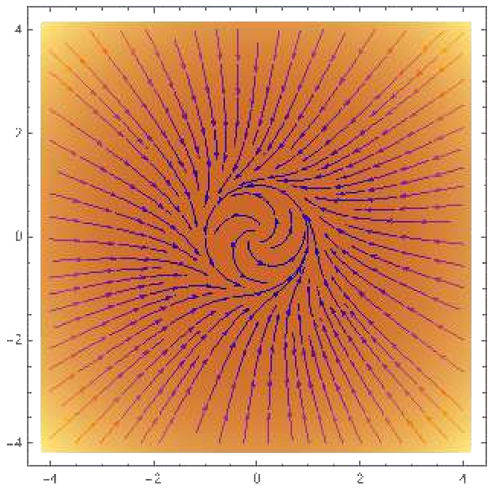

Figure 2.3: Phase portrait after slicing the sphere as the planar system

Now to understand the whole phenomenon throughout the sphere , observe the illustration in figure-2.4

2.7 Strange Attractors

stranger attractors appear in the various fields of mathematical sciences and biological sciences as well . They are very isolated in nature in terms of they are very irregular, non periodic , chaotic in time evolution .
Strange attractors consists of infinitely many points in a finite dimensional space (not necessarily euclidean) . These are extremely abstract mathematical object but still a little bit of computer programming has shown a new path with its intense light .

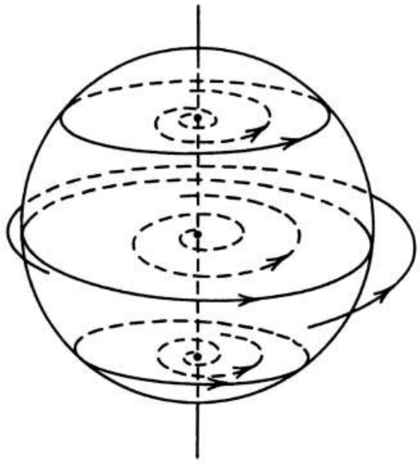

Figure 2.4: Invariant flow on the sphere

(a) Chaotic atmosphere in Jupiter

(b) Fumes generated from an ignited cigarette

Figure 2.5: Two different kinds of chaotic behaviours

2.8 Description of the time evolution of the dynamical system

When we mention the ‘dynamical system’ , we mean physical , biological and chemical system with the variational parameters $x_1, x_2, \dots x_m$[3] .
Consider the parameters vary with the time , $x_1(t), x_2(t), \dots, x_m(t)$. Note that t can intake any value . But for the sake of simplicity we first consider $t \in \mathbb{Z}_+$.

[3]In a physical , biological or chemical system there exist too many factors, we call them parameters

Now specifying the parameters $x_1, x_2, \dots x_m$ in time $t+1$, $\forall\ t \in \mathbb{Z}_+$.

$$x_1(t+1) = \phi_1(x_1, x_2, \dots, x_m) \tag{2.18}$$
$$x_2(t+1) = \phi_2(x_1, x_2, \dots, x_m) \tag{2.19}$$
$$\dots \tag{2.20}$$
$$\dots \tag{2.21}$$
$$x_m(t+1) = \phi_m(x_1, x_2, \dots, x_m) \tag{2.22}$$

where x_i is a smooth function , $\forall\ i = 1, 2, 3, \dots m$. Now suppose at $t = 0$, we successively have $x_1(0), x_2(0), \dots, x_m(0)$. Then easily we may obtain the values of the parameters as the time evolutes . Clearly this time evolution defines a discrete-dynamical system .

2.9 Definition

Let M be an m-dimensional smooth boundaryless compact manifold and define a map $\Phi : M \to M$ by

$$\Phi(x_1, x_2, \dots, x_m) = (\phi_1(x_1, x_2, \dots, x_m), \phi_2(x_1, x_2, \dots, x_m), \dots, \phi_m(x_1, x_2, \dots, x_m)) \tag{2.23}$$

We say a bounded set A in the m-dimensional manifold is a 'strange attractor' if there exists an open neighborhood satisfying the properties given below :

1. $dim\, U = m$ and $A \subset U$. That is for each point of M there exits a ball centered at A and entirely contained in U .
2. For each initial point $x_0 \in U$, all the co-ordinates $(x_1(t), x_2(t), \dots, x_m(t))$ remains in $U\ \forall\ t > 0$. For large t , the flows tends to come close to A where A is an attracting set .
3. There also exists a sensitive , Sharp dependence on the initial condition $x_0 \in U$.

2.9.1 Lorenz attractor

From the work of Mathematician and meteorologist Edward Lorenz in 1963 , mathematician Sparrow indicates for certain values of the parameters σ , ρ and β , the system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$

has a strange attracting set . For example choose $\sigma = 10$, $\rho = 28$, $\beta = \frac{8}{3}$ and obtain the figure given below of Lorenz attractor

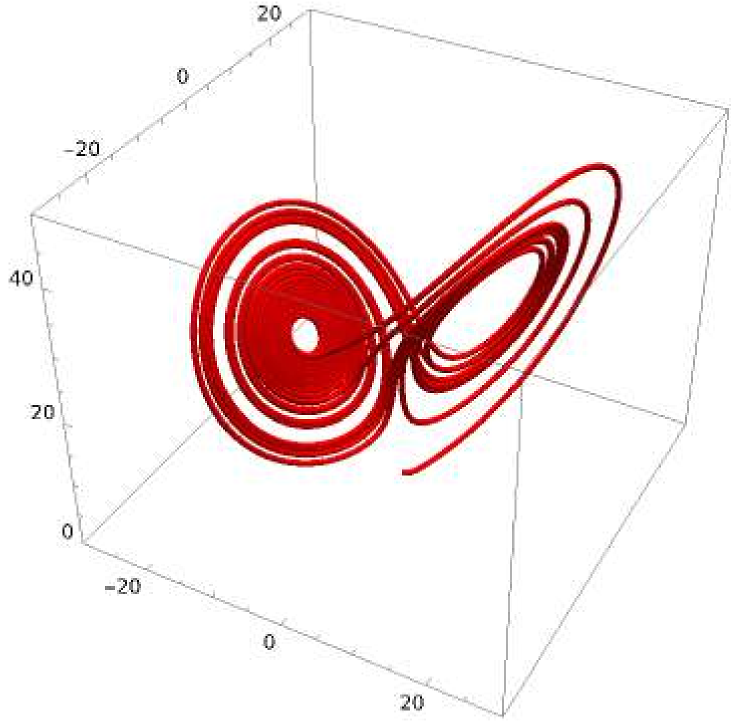

Figure 2.6: Lorenz attractor for the meteorological system

CAVEAT

One should be very careful while dealing with a strange attractors . As we mentioned before that these are very initial condition sensitive,unusual, irregular physical, biological and chemical phenomenon , so it cannot be treated in a casual manner . Just have a look upon the python generated picture below

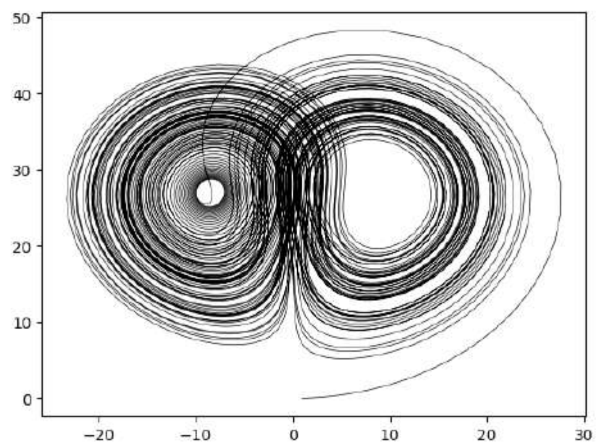

Figure 2.7: Strange attractor in a planar outlook

One might be thinking that the trajectories are intersecting .This is the very point we must be alert that unfortunately and interestingly these are not intersecting at all . These strange behaviour makes the dynamical system so unique !

2.9.2 Rösseler system

The Rösseler attractor system system of three non linear ordinary differential equations given by

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = bx - cz + xz$$

Röseller attractor was intended to to behave similarly to the Lorenz attractor but simpler and has only one manifold . An orbit within the except follows and outward spiral close to the xy-plane around the unstable fixed point . Rösseler studied the strange attractor by choosing the parameters as

$$a = 0.2$$
$$b = 0.2$$
$$c = 5.7$$

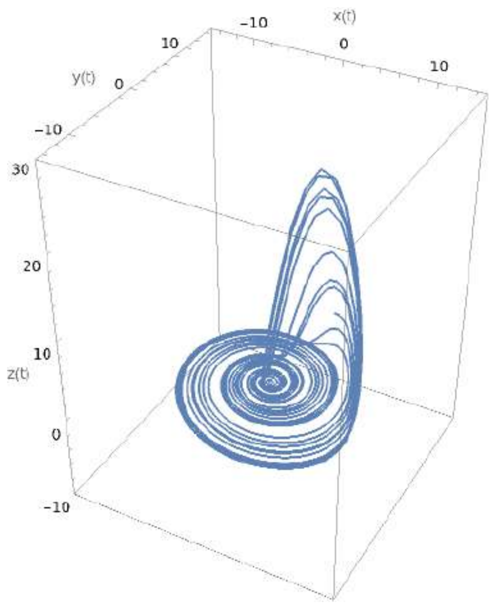

Figure 2.8: Rösseler system

2.9.3 Reinjection Principle and Shil'nikov Criterion

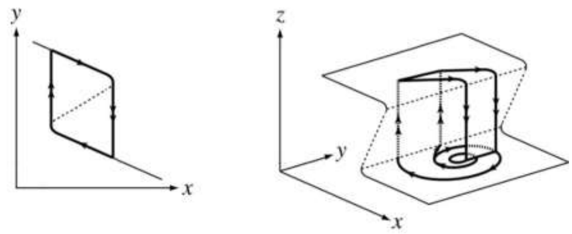

Figure 2.9: Reinjection and Z-shaped slow manifold in both $\mathbb{R}^2$ and $\mathbb{R}^3$

Rösseler was inspired by the geometry of flows in three dimension and in particular by reinjection principle which is based on the feature of relaxation types systems to often present Z-shaped slow manifold in their phase space
The Shil'nikov criterion states that *"reinjection is faster than the spiraling-out motion* .
To verify the statement we need to choose the parameters here as

$$a = 0.38$$
$$b = 0.3$$
$$c = 4.820$$

And hence obtain

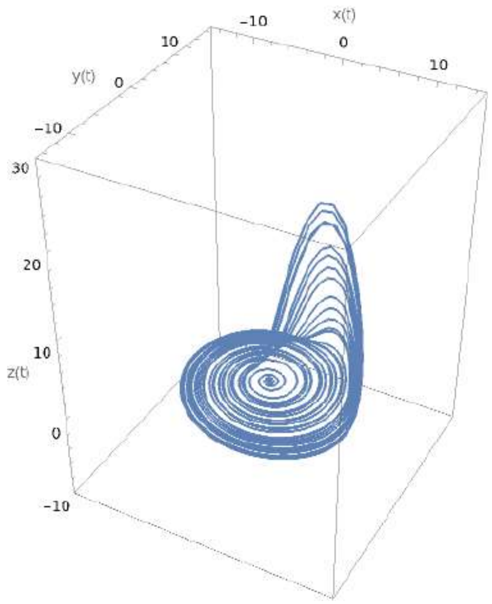

Figure 2.10: Phase portrait after introducing new variables

Chapter 3

Stability Of Limit Cycles

3.1 Historical Background

Limit cycles in planar differential systems commonly occur in the domain of modelling both the technology and natural sciences . The theory of limit cycles is always a field of shocking and interesting phenomenons . The study of living cycles first began during 18^{th} century with hand of Henri Poincare . He was the first mathematician to use limit cycles to solve problem of control of hydraulic machines . Until now historiographically it is considered that the mathematician Andronov(1929) was the first man to find a correspondence between the periodic solutions of self-sustaining dissapative system and the limit cycles introduced by Poincare . Most of the early history in the theory of limit cycles in the plane , initiated by the both physicist and mathematicians stimulated by practical problem .

3.2 Definition

A *'limit cycle'* is trajectory L of a planar system (2.1) which is either the ω-limit set or α-limit set of another trajectory Γ(i.e $\alpha(\Gamma)$ and $\omega(\Gamma)$) other than L so that L is closed and isolated that is to say there does not exist any closed trajectory as L in any neighborhood of L .

Few comprehensive examples of limit cycle are given below

3.2.1 Oscillation of Violin string

[1] Consider the dynamical system given as

$$\ddot{x} + \epsilon(\frac{1}{3}\dot{x}^2 - 1)\dot{x} + x = 0 \tag{3.1}$$

This equation can be represented as the planar autonomous first order ordinary differential equation

[1]This example was first introduced by physicist Rayleigh in 1877

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon(\frac{y^2}{3} - 1)y \end{aligned} \tag{3.2}$$

A phase portrait is illustrated in the figure (3.1) ,

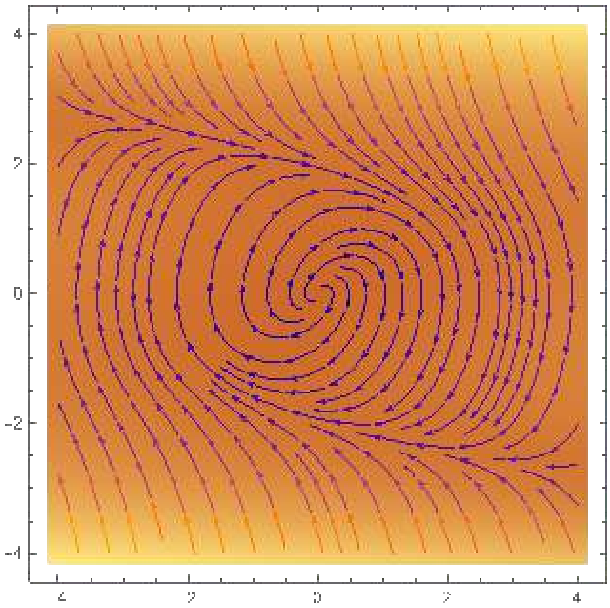

Figure 3.1: Periodic behaviour of Rayleigh system for $\epsilon = 1.0$

3.2.2 Example(A formal way to study limit cycles)

consider the dynamical system in polar co-ordinates as

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1 \end{aligned} \tag{3.3}$$

Let r^* the equilibrium points of the system . Then we have ,

$$r^*(1 - r^{*2}) = 0 \tag{3.4}$$

The only equilibrium points are thus obtained by $r^* = 0$ and $r^* = 1$. To find the stability of the fixed points , have a look on the diagram shown in figure 3.2 .

From the above diagram the figure clearly $r^* = 0$ and $r^* = 1$ are stable and unstable equilibrium points respectively.

Now for the illustration of the limit cycle observe that $r^* = 0$ is unstable. So , the α-limit set will repel from the origin and meet the circle $r^* = 1$ and since $r^* = 1$ is stable , all the trajectories will be attracted to the circle as ω-limit set.

To study the dynamical system in a descriptive manner we study the system in Cartesian co-ordinates instead for the sake of convenience in Mathematica .

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

Now putting

$$x\dot{x} + y\dot{y} = r\dot{r} = r^2(1 - r^2) \tag{3.5}$$

Or

$$x\dot{x} + y\dot{y} = (x^2 + y^2)(1 - x^2 - y^2) \tag{3.6}$$

Again similarly

$$x\dot{x} - y\dot{y} = (x^2 + y^2) \tag{3.7}$$

Thus solving the equations (3.5) and (3.7) for $\dot{x}$ and $\dot{y}$ one should obtain the first order non-linear autonomous planar system of non-linear differential equations[2]

$$\dot{x} = x(1 - x^2 - y^2) - y$$
$$\dot{y} = y(1 - x^2 - y^2) - x$$

To obtain the equilibrium points equalize the equations to zero i.e.

$$\dot{x} = x(1 - x^2 - y^2) - y = 0$$
$$\dot{y} = y(1 - x^2 - y^2) - x = 0$$

Hence the phase portrait is obtained and shown in the figure 3.2

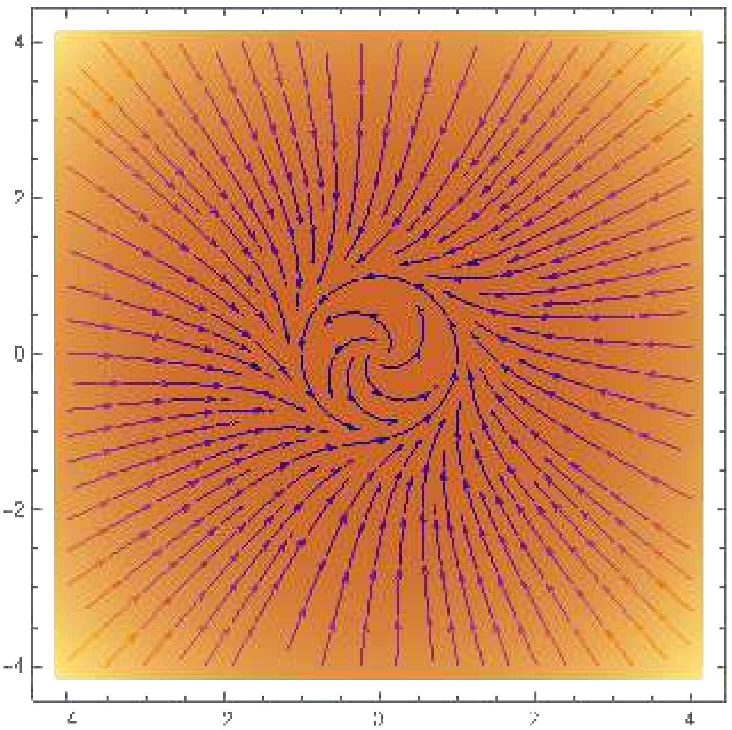

Figure 3.2: The phase portrait of the above system

[2]In the second example one can notice that the limit cycle is an ω-limit set

3.2.3 Example(A similar problem to example 3.2.2)

The system of differential equations is given in polar coordinates

$$\begin{aligned} \dot{r} &= r(r^2 - 1) \\ \dot{\theta} &= 1 \end{aligned} \tag{3.8}$$

By similar attempts as done in the second example one should obtain the system of equations

$$\begin{aligned} \dot{x} &= x(x^2 + y^2 - 1) - y \\ \dot{y} &= y(x^2 + y^2 - 1) - x \end{aligned}$$

By the similar arguments above it can be said that the origin is a stable fixed point . But here $r^* = 1$ is unstable .

3.2.4 A description of the important features of the following system of polar differential equations

Consider

$$\begin{aligned} \dot{r} &= r(1 - r)(2 - r)(3 - r) \\ \dot{\theta} &= -1 \end{aligned} \tag{3.9}$$

Solution :
Observe that the system has a unique critical point at the origin since $\dot{\theta}$ is non-zero. Again by the observation there are three limit cycles of the radii r=1,2,3 respectively , call them $r_i \; \forall \; i = 1, 2, 3$ and all them are centered at the origin (see figure 3.3).

Now one can easily study the features from the given figure above . There is a critical point at the origin . Any trajectory starting at this point remains here forever .

Also note that we have total 3 limit cycles here , call them $\Gamma_i, \forall i = 1, 2, 3$. The trajectories starting at (1,0) reaches (-1,0) whenever $t_1 = \pi$. Then again the trajectory begins at (-1,0) and accomplishes its journey (1,0) in $t_2 = \pi$.
Thus we obtain a periodic cycle of period 2π . Now choose the points $P = (\frac{1}{2}, 0)$ and $Q = (4, 0)$ on the plane .

We are interested in the ω-limit set and α-limit set corresponding to the points.If one look closely may surely observe that the trajectories are coiling the point $P = (\frac{1}{2}, 0)$. That is the flows of the vector field have a tend to move through P for indefinite time . Hence , $\omega(P) = \Gamma_1$.
Again , since every trajectory is getting attracted towards the cycle there is no such trajectory that is being repelled from the cycle $\alpha(P) = (0, 0)$.
So , now by similar arguments, ω-limit set for the point Q thus given by $\omega(Q) = \Gamma_3$, all the trajectories from infinity are attracted to the cycle whereas see that $\alpha(Q) = \infty$.

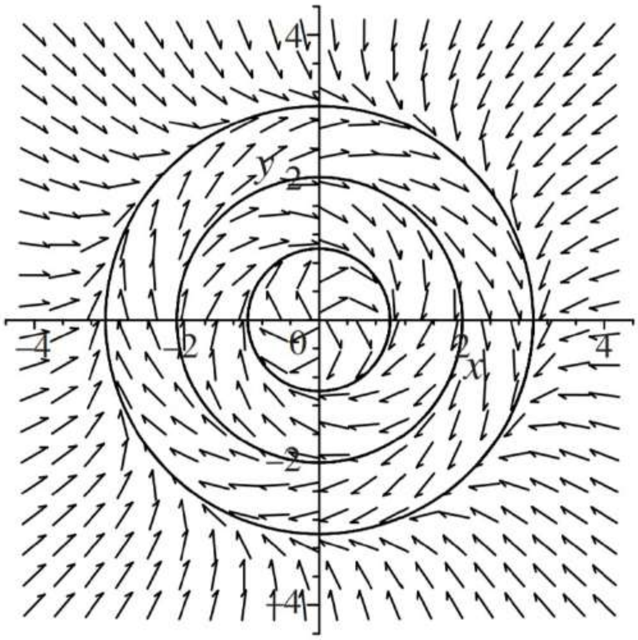

Figure 3.3: A rough phase portrait

Thus , the annulus , $A_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2, r \in (0, 1)\}$ is positively invariant that is the flow is not coming outside of the cycle and the annulus $A_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2, r \in (1, 2)\}$ is negatively invariant[3]

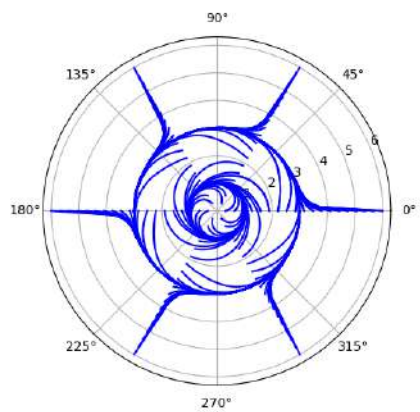

Figure 3.4: Phase portrait

[3]Note that positive and negative is completely determined by seeing whether it is being attracted or repelled from the cycle

3.3 Discussion on stability

So far a few examples of limit sets , attracting sets , attractors and also the limit cycles have been observed. But no word for stability of is uttered. but the question of stability must arise and that is why few useful definitions are given below that will help us to find stability of an orbit .

1. **Stable Limit Cycles**

   Let Γ be a closed trajectory. If for a sufficiently small neighbour (outer or inner) of Γ exists such that all the trajectories in it are non-closed and they take Γ as ω-limit sets . Then , Γ is called an (externally or internally) stable limit cycle .

2. **Unstable limit cycles**
   Let Γ be a closed trajectory. If for a sufficiently small neighbour (outer or inner) of Γ exists such that all the trajectories in it are non-closed and they take Γ as α-limit sets . Then , Γ is called an (externally or internally) unstable limit cycle .

3. **Semi-stable Limit Cylces**
   A closed isolated trajectory is called the semitable or half stable limit cycle if it is externally or internally stable and is internally or externally unstable .

4. **Compound Limit Cycles**

   A limit cycle is called a compound limit cycle if it does not belong to any of the above 3 cases .

3.3.1 Example

Consider the example 3.2.2 .
We now analyse the stability of the system by zooming the phase portrait little bit of it. As we there stated that origin is an unstable fixed point and hence we see in the Figure-3.5 that the trajectories are leaving the origin but showing a trend to approach the circle $r = 1$ after a certain time .
Also Observe that all the trajectories outside the circle are approaching the circle $r = 1$ and hence these points represents ω-limit set . So, this is a stable limit cycle . Also note that to look more closely we reduced the range from $(-4, 4)$ to $(-2, 2)$ and thus we obtain a new phase-portrait which is more vivid .

Remarks

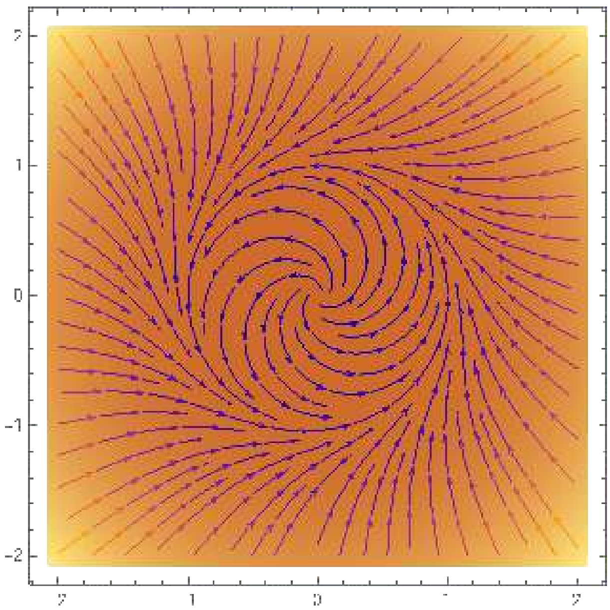

Figure 3.5: A closer outlook of the system

1. With this example above it is also explicit what we actually meant by a *trajectory* or an *orbit* Γ of the given system is nothing but the equivalence class of solution curves $\phi(*, x)$ with $x \in \Gamma$

2. We can also develop the qualitative study by expanding or contracting the range and certain variables.

3.3.2 Example

Take the example as gave in (3.2.4) where we had three limit cycles $\Gamma_1, \Gamma_2, \Gamma_3$. The only fixed points of the system are $r^* = 0, 1, 2, 3$ respectively where
$r^* = 0$ is unstable , $r^* = 1$ is stable , $r^* = 2$ is unstable and finally $r^* = 3$ is stable
Now comparing our results with the given figure we obtain Γ_1 is a stable limit cycle whereas all the trajectories are getting repelled from Γ_2 . Hence it provides an α-limit set and hence Γ_2 is an unstable limit cycle.
On the other hand from the given figure it is also clear that Γ_3 is stable limit cycle . Importantly $\Gamma_1, \Gamma_2, \Gamma_3$ are all the periodic orbits of period 2π that can be verified by integrating $\dot{\theta} = -1$ with respect to the time t .

3.3.3 Example

Consider the system

$$\dot{x} = -y + x(x^2 + y^2) \sin \frac{1}{\sqrt{(x^2+y^2)}} \tag{3.10}$$

$$\dot{y} = x + y(x^2 + y^2) \sin \frac{1}{\sqrt{(x^2+y^2)}} \tag{3.11}$$

where $x^2 + y^2 \neq 0$
Evidently , this is a smooth vector field on $\mathbb{R}^2$.
We try to rewrite the system in polar coordinates by considering an arbitrary complex number $z = x + iy$.

$$\begin{aligned}\dot{z} &= \dot{x} + i\dot{y} \\ &= -y + x(x^2+y^2)\sin\frac{1}{\sqrt{(x^2+y^2)}} + i[x + y(x^2+y^2)\sin\frac{1}{\sqrt{(x^2+y^2)}}] \\ &= i(x+iy) + (x^2+y^2)\sin\frac{1}{\sqrt{(x^2+y^2)}}(x+iy) \\ &= iz + z|z^2|\sin\frac{1}{z}\end{aligned}$$

Hence

$$\dot{z} = iz + z|z^2|\sin\frac{1}{z} \tag{3.12}$$

Now let $z = re^{i\theta}$ and putting in (3.12)

$$\begin{aligned}\dot{z} &= iz + z|z^2|\sin\frac{1}{z} \\ &= ire^{i\theta} + r^3e^{i\theta}\sin\frac{1}{r}\end{aligned}$$

Thus

$$\dot{z} = ire^{i\theta} + r^3e^{i\theta}\sin\frac{1}{r} \tag{3.13}$$

Also we have

$$\dot{z} = ire^{i\theta}\dot{\theta} + \dot{r}e^{i\theta} \tag{3.14}$$

Now equalization of (3.13) and (3.14) thus provides us the new equivalent system in polar coordinates

$$\begin{aligned}\dot{r} &= r^3\sin\frac{1}{r} \\ \dot{\theta} &= 1\end{aligned}$$

Now observe that $r^* = 0$ and $r^* = \frac{1}{n\pi}$, $\forall\, n \in \mathbb{Z}_+$ are the only fixed points of the polar system where $\dot{r} = 0$. To understand the above mentioned words it is necessary to look it more closely.
Note that if $r_n = \frac{1}{n\pi}$, $\lim\limits_{n\to\infty} r_n = 0$ that is the limit cycles accumulate at the neighbourhood of the origin . Now let Γ_n defined as whenever $r = r_n$, $\forall\, n \in \mathbb{Z}_+$ be the limit cycles such that $\lim\limits_{n\to\infty} d(\Gamma_n, 0) = 0$ holds .
Evidently Γ_{2n} is stable and Γ_n is unstable .

For more accuracy and comprehending the accumulation see figure 3.7 .

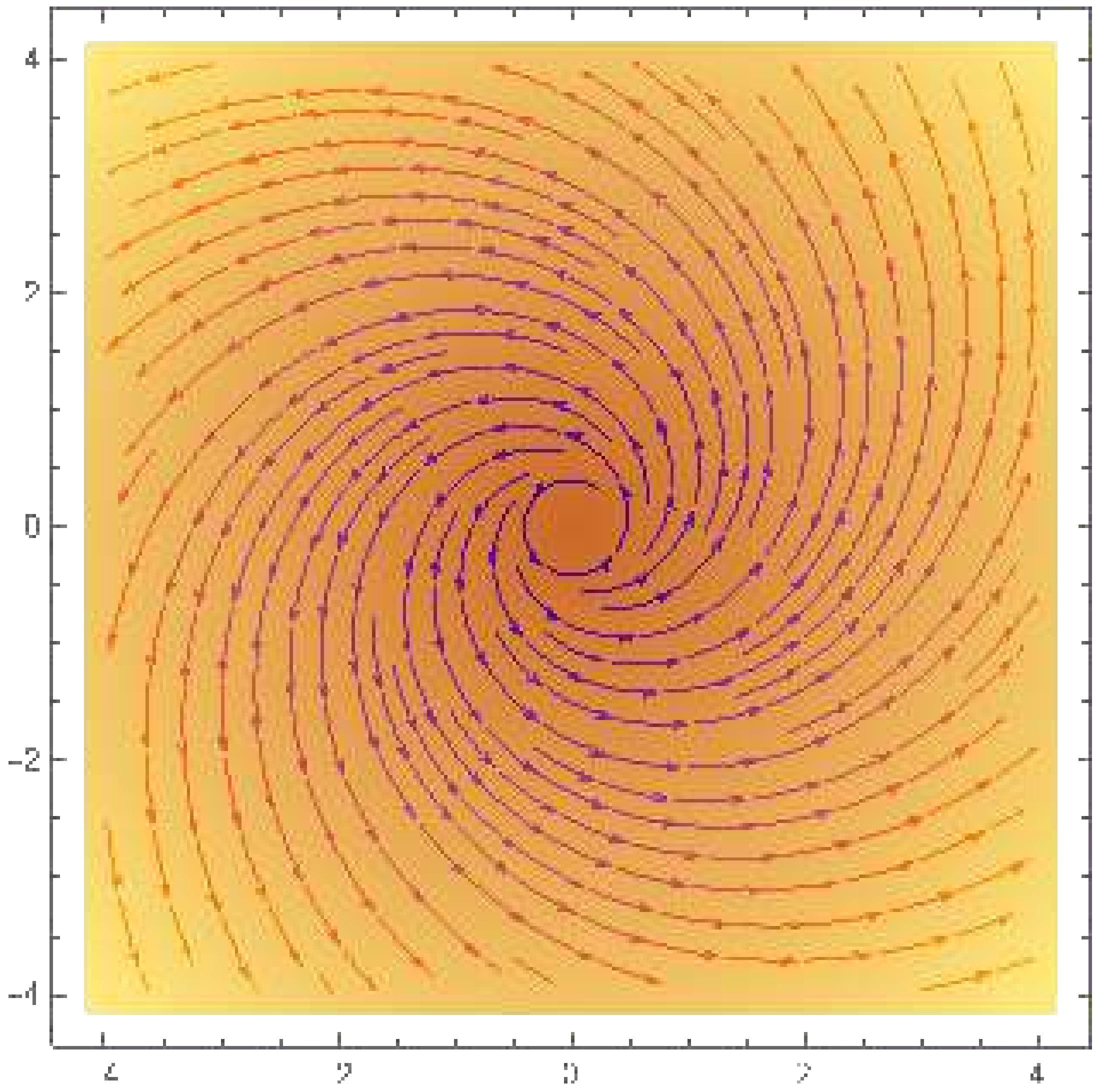

Figure 3.6: Accumulation takes place

3.4 Stability of manifolds

We earlier defined the manifolds very briefly and gave few examples and described the differentiable manifold in a very short essay obviously. Now one might not shock if one asks about the stability of manifolds as well . We will elaborate it later on .
Roughly speaking most of our description will be upon checking local diffeomorphism . That is the phase portraits are to be locally diffeomorphic .

First get the facts what does it mean by 'local diffeomorphism'. Take this example first

3.4.1 Example

Define the smooth map $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = e^t$.
Let $t \in \mathbb{R}$, $df|_t : \mathbb{R} \to \mathbb{R}$ be defined as ,

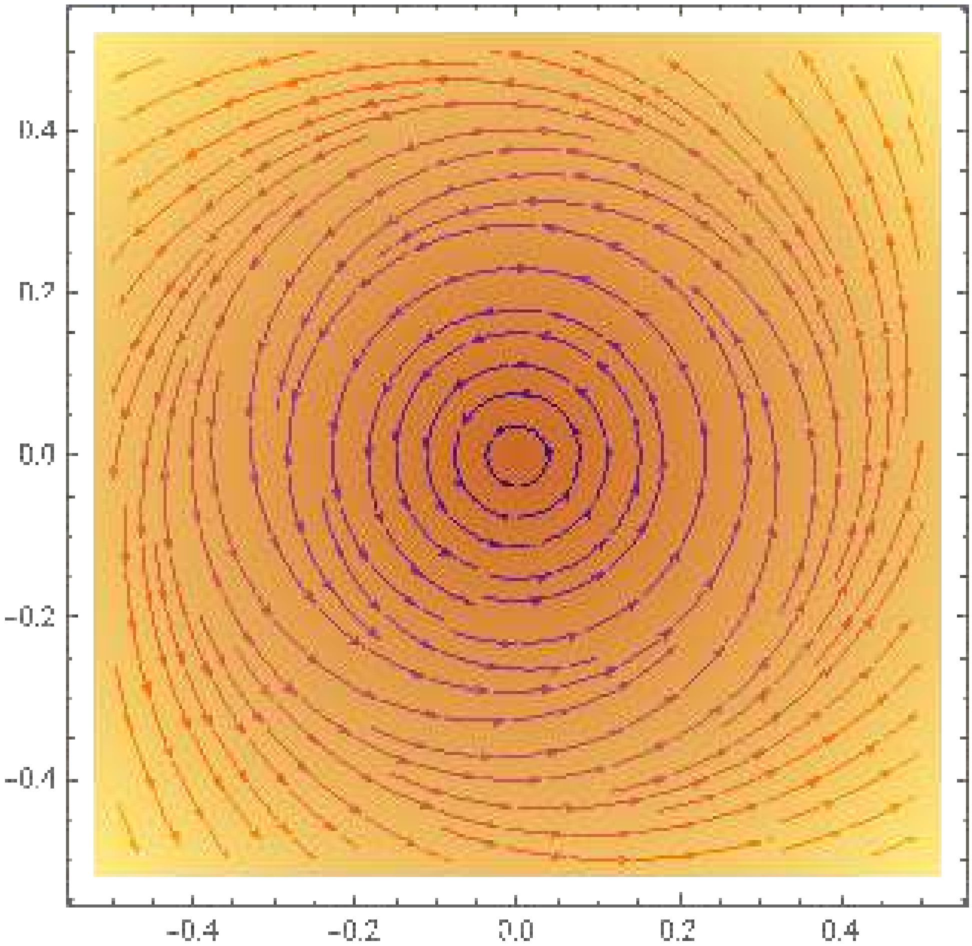

Figure 3.7: The accumulation of Γ_n is little bit vivid here

$df|_t(s) = e^t s \ \forall s \in \mathbb{R}$.
Now note that $e^t \neq 0 \implies ker\ df|_t = \{0\} \implies rank\ df|_t = 1$
Hence , $df|_t$ is bijective .
So , f is a local diffeomorphism . But note that $f^{-1}(0) = \phi$. Hence f is not a diffeomorphism. [4]

3.4.2 Example

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(t_1, t_2) = (e^{t_1}\cos t_2, e^{t_1}\sin t_2)$, $\forall \ (t_1, t_2) \in \mathbb{R}^2$.
Clearly f is smooth . Consider the derivative map
$df|_t : \mathbb{R}^2 \to \mathbb{R}^2$ which is a linear transformation and hence the matrix(Jacobian) thus obtained by

[4]Let M and N be two smooth manifolds of dimensions m and n respectively . Let $f : M \to N$ be a smooth map ($M \subseteq \mathbb{R}^K \& N \subseteq \mathbb{R}^L$) . Then take the derivative map ,
$df|_p : T_pM \to T_qN$, where $q = f(p)$. Hence we obtain a linear transformation .
Then f is said to be a local diffeomorphism at p if , $df|_p$ is bijective .

$Jf(t) = \begin{pmatrix} e^{t_1}\cos t_2 & e^{t_1}\sin t_2 \\ -e^{t_1}\sin t_2 & e^{t_1}\cos t_2 \end{pmatrix}$
Now compute $detJf(t) = e^t \neq 0$
$\implies ker\ df|_t = \phi$
$\implies rank\ df|_t = 2$. So $df|_t$ is bijective . Hence f is a local diffeomorphism at $t \in \mathbb{R}^2$.
But note that f is not injective . So f is not a diffeomorphism.

3.5 Topological Conjugacy

Define a dynamical endomorphism $f : U \to U$ (piecewise continuous) where $U \subset X$ and X be a complete separable metric space .
Also let $g : V \to V$ be another dynamical endomorphism and both U & V are open in X .
Then , f and g are said to be to *topologically conjugate* we write $f \xrightarrow{TopCon} g$ if there exist a smooth map $\phi : U \to V$ so that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ f\downarrow & & \downarrow g \\ U & \xrightarrow{\phi} & V \end{array}$$

commutes .

3.5.1 Symbolic systems and endomorphisms

Without any loss of generality we move to the category of topological spaces **Top** so that all the morphisms are continuous functions . We construct a class of topological models called the *symbolic systems* . Let $J_N = \{0, 1, 2, 3, \dots, N\}$ and consider
$H^\infty_{\mathbb{Z}\#} := \{x = \{x_0, x_1, x_2, \dots\} : x_i \in J_N\}$. Now define a metric d on $H^\infty_{\mathbb{Z}\#}$ by

$$\rho(x, y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^k}$$

where $y \in H^\infty_{\mathbb{Z}\#}$.
It can be verified that the pair $(H^\infty_{\mathbb{Z}\#}, \rho)$ is a compact metric space .
Thus define a left shift operator (compact operator) on $H^\infty_{\mathbb{Z}\#}$, $\mathcal{L} : H^\infty_{\mathbb{Z}\#} \to H^\infty_{\mathbb{Z}\#}$ by
$\mathcal{L}(\{x_0, x_1, x_2, \dots\}) = (\{x_1, x_2, \dots\})$
Also note that $\mathcal{L} \in End(H^\infty_{\mathbb{Z}\#})$, endomorphisms on $H^\infty_{\mathbb{Z}\#}$

Stable manifold theorem

Now we enter into the topic related to the stability of manifolds .First consider the smooth vector field

$$\dot{x} = f(x), f \in C^{\infty}(U) \tag{3.15}$$

where U being any open in $\mathbb{R}^n$. To define a hyperbolic fixed point consider the smooth map.[5] Now take the linear transformation
$df|_t : T_pU \to T_p\mathbb{R}^n$ that is $df|_t : \mathbb{R}^n \to \mathbb{R}^n$ which is a linear transformation.
Hence its Jacobian exists since $\mathbb{R}^n$ is a finite dimensional Hilbert space .[6] Therefore the Jacobian $Jf(p)$ exists .
If $Jf(p)$ has no eigen values on the unit circle , then p is called a hyperbolic fixed point. Alternatively, an equilibrium point x^* of a smooth vector field $\dot{x} = f(x)$ is said to be a hyperbolic fixed point if all the eigen values of the Jacobian $Jf(x^*)$ have the non-zero real part.
Although the two above definitions does not seem to be very obvious in nature but they are equivalent . Two verify the equivalency of the definitions we introduce new topic called the ***Gershgorin Disk*** . We denote in a ad-hoc way by

$$\mathbb{D}_i^{GSR} = \{z \in \mathbb{R}^2 : |z - A_{ii}| \leq r_i\} \tag{3.16}$$

where A_{ii} is the leading diagonal element of the i^{th}-row of a complex square matrix A and r_i is the radius of disk defined above .
But the question is that how does this disc work ? Roughly speaking , this extraordinary disk guarantees whether a value on the xy-plane or on the argand plane is really an eigen value of the given matrix A or not . Take the example first

3.5.2

Consider the matrix $A = \begin{pmatrix} 1+2i & 1 \\ 2i & -3 \end{pmatrix}$. Now to construct a Gershgorin disk and compute
$\rho_i(A) :=$ Sum of the absolute value of the entries of the i^{th}- row and also $\rho(A) = max\{\rho_i(A) : 1 \leq i \leq n\}$.
Then the radius of the disk is defined by

$$r_i = \rho_i(A) - |A_{ii}| \tag{3.17}$$

[5]**Remarks :**

1. Note that we could similarly define it for an abstract manifold of finite dimensions but working with Euclidean space and its induced standard topology is bit easier to grasp .
2. similar attempts could be done for the infinite dimensional manifolds but in those cases topological boundary becomes important and the embedding would be difficult for us to comply

[6]p must be a fixed point of f

So we have

$$\begin{aligned} r_1 &= \rho_1 - |A_{11}| \\ &= (\sqrt{5}+1) - \sqrt{5} \\ &= 1 \end{aligned}$$

Then the first disk , $\mathbb{D}_1^{GSR}$ is centered at $1+2i$ and radius 1 . Similarly we move to the second disk $\mathbb{D}_2^{GSR}$ which is centered at -3 with radius 2 since

$$\begin{aligned} r_2 &= \rho_2(A) - |A_{22}| \\ &= 2+3-3 = 2 \end{aligned}$$

Now if λ_1 and λ_2 be two eigen values of the matrix A then $\lambda_1, \lambda_2 \in \mathbb{D}_1^{GSR} \cup \mathbb{D}_2^{GSR}$. From the geometry in argand plane , it can be observed that in the definition of the hyperbolic fixed point above the eigen values those which are not on the unit circle[7].

3.5.3 Example

Let us consider the non-linear dynamical system

$$\begin{aligned} \dot{x} &= x - xy \\ \dot{y} &= -y + xy \end{aligned}$$

Fixed points are obtained as $(0,0)$ and $(1,1)$ respectively.The Jacobian
$Jf(x,y) = \begin{pmatrix} \text{1-y} & -x \\ y & -1+x \end{pmatrix}$. At the points $(0,0)$ and $(1,1)$ the Jacobians are thus obtained by
$A = Jf(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = Jf(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The eigen values of these two above matrices are respectively ± 1 and $\pm i$. Now define two Gershgorin disks

$$\mathbb{R}D_i^{GSR} = \{x \in \mathbb{R}^2 : |x - A_{ii}| \le r_i\} \tag{3.18}$$
$$\mathbb{C}D_j^{GSR} = \{x \in \mathbb{C} : |z - B_{jj}| \le r_j\} \tag{3.19}$$

Consider the disk (3.18) first . Here we have two disks $\mathbb{R}D_1^{GSR} = \{x \in \mathbb{R}^2 : |x-1| \le 0\}$

[7]One must be careful of the notion that the Gershgorin circle theorem does not guarantee whether the values inside the disc are eigen values of the matrix , it rather guarantees that the values which are not inside the disks are undoubtedly not the eigenvalues of the matrix

and $\mathbb{R}D_2^{GSR} = \in \mathbb{R}^2 : |x+1| \leq 0\}$. Collectively , $\mathbb{R}D_i^{GSR} = \mathbb{R}D_1^{GSR} \cup \mathbb{R}D_2^{GSR} = \{1, -1\}$ and its boundary is empty along with all the eigen values are on the real axis . So , the fixed point $(0,0)$ is a hyperbolic fixed point .
Now again , take the disk (3.19)

$\mathbb{C}D_1^{GSR} = \{z \in \mathbb{C} : |z-0| \leq 1\}$ and $\mathbb{C}D_2^{GSR} = \{z \in \mathbb{C} : |z-0| \leq 1\}$. Hence collectively ,
$\mathbb{C}D_i^{GSR} = \mathbb{C}D_1^{GSR} \cup \mathbb{C}D_2^{GSR} = \{z \in \mathbb{C} : |z| \leq 1\}$. Hence the boundary $\partial \mathbb{C}D_i^{GSR} = \{z \in \mathbb{C} : |z| = 1\}$ which evidently contains the eigen values $\pm i$. So again by Gershgorin circle theorem $(1,1)$ is a non-hyperbolic fixed point .

Thus the above example justifies the original and the alternative definitions of a hyperbolic fixed point .

3.6 Hartman-Großman theorem

let p be a hyperbolic fixed point of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$. Then for an open neighbourhood U in $\mathbb{R}^n$ of p , f is topologically conjugate to its linearization $df|_p$.

3.6.1 Statement of the stable manifold theorem

consider the given system

$$\dot{x} = f(x) \tag{3.20}$$

Let $Jf(x)$ be the Jacobian of the smooth map . Suppose that the system has an isolated hyperbolic equilibrium point at the origin . And the Jacobian $Jf(0)$ has k eigen values having negative real part and $(n-k)$ eigen values having positive real part .

1. Then there exists a k-dimensional differentiable manifold say W^s which is
   (a) tangent to the stable generalised eigen space E^s of the linearized system $\dot{x} = Jf(0)x$ at the origin .
   (b) is invariant with respect to its flow
   (c) For all initial conditions $x_0 \in W^s$, $\lim_{t\to\infty} \phi(x_0, t) = 0$. That is to say $0 \in \omega(\Gamma)$, for some trajectory Γ
2. There exist an $(n-k)$-dimensional differentiable manifold say W^u which is
   (a) tangent to the unstable generalised eigen space E^u of the linearized system $\dot{x} = Jf(0)x$ at the origin .
   (b) is invariant with respect to its flow
   (c) For all initial conditions $x_0 \in W^s$, $\lim_{t\to-\infty} \phi(x_0, t) = 0$. That is to say $0 \in \alpha(\Gamma)$, for some trajectory Γ

3.6.2 Verification of the Hartman-Großman theorem for the given system and also finding the stable and unstable manifolds

Consider the system

$$\dot{x} = -x \text{ and } \dot{y} = x^2 + y \tag{3.21}$$

Solution

The equations in the above system can be solved exactly at a time t .
Solving the first equation of the given system we obtain

$$x = x_0 e^{-t} \tag{3.22}$$

Now substitute the value of x from (3.22) in the second equation of the given system

$$\dot{y} = x_0^2 e^{-2t} + y$$
$$\implies \frac{dy}{dt} - y = x_0^2 e^{-2t}$$
$$\implies y.e^{-t} = \int x_0^2 e^{-3t} + K$$

Initially at $t = 0$ we have $y = y_0$. So

$$K = y_0^2 + \tfrac{1}{3} x_0^2$$
$$\implies ye^{-t} = -\tfrac{1}{3} + x_0^2 e^{-3t} y_0 + \tfrac{1}{3} x_0^2$$
$$\implies ye^{-t} - y_0 = \tfrac{x_0 2}{3}(1 - e^{3t})$$

Therefore ,

$$y = y_0 e^t + \tfrac{1}{3} x_0^2 (e^t - e^{-2t}) \tag{3.23}$$

Thus the flow is obtained in the form of a parametrized curve as

$$x(t) = x_0 e^t \tag{3.24}$$
$$y(t) = y_0 e^t + \tfrac{1}{3} x_0^2 (e^t - e^{-2t}) \tag{3.25}$$

Rewriting the equations again,we have

$$x(t) = x_0 e^t \tag{3.26}$$
$$y(t) = (y_0 + \tfrac{1}{3} x_0^2) e^t - \tfrac{1}{3} x_0^2 e^{-2t} \tag{3.27}$$

Since the origin is the only equilibrium point it is isolated . now to verify the hyperbolicity write down the jacobian

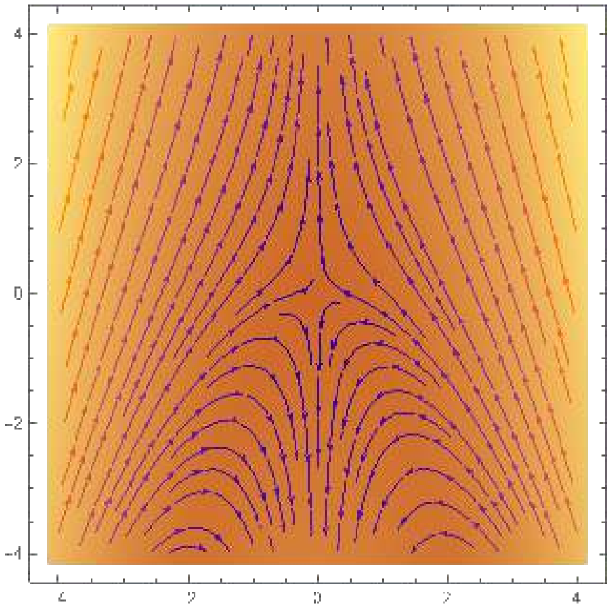

Figure 3.8: Phase portrait of the given system

$Jf(x,y) = \begin{pmatrix} \text{-1} & 0 \\ 2x & 1 \end{pmatrix}$ At the origin , $Jf(0,0) = \begin{pmatrix} \text{-1} & 0 \\ 0 & 1 \end{pmatrix}$ Hence ± 1 are the eigen values and hence $(0,0)$ is a hyperbolic equilibrium point .
Thus the phase portrait is provided in the figure 3.8 .

Linearization of the given system

To apply the Hartman-Grroßman theorem we need the linearization of the given smooth vectot field defined in (3.21) . Thus the linear approximation of the given system can be expressed as

$$\dot{x} = -x \text{ and } \dot{y} = y \tag{3.28}$$

The solution curve in the form of parametrized curve obtained as

$$x = x_0 e^{-t} \text{ and } y = y_0 e^{t} \tag{3.29}$$

Now using mathematica one must obtain the phase portrait as given in figure 3.9 .
So by Hartman-Großman theorem we conclude that the figure 3.8 and figure 3.9 are topologically conjugate and there should be a local diffeomorphism between them .
Now the only task is left so far to find the stable and unstable manifolds. One thing is to be remembered that the stable or unstable manifold is dependent upon the initial condition (x_0, y_0) so that it causes to approach the flow to the origin as $t \to \infty$.
In the expression of $x(t)$, $\lim_{t\to\infty} x_0 e^{-t} = 0$ i.e. regardless of our choice of x_0. That is to be precised literally have no restrictions on the choice of initial conditions .

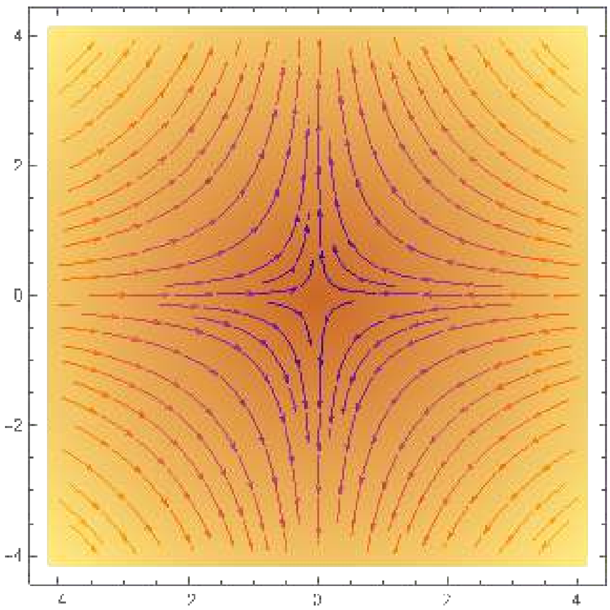

Figure 3.9: Phase portrait of the linearized system

Now the relation

$$\lim_{t\to\infty}[(y_0+\tfrac{1}{3}x_0^2)e^t-\tfrac{1}{3}x_0^2e^{-2t}]=0$$

must be satisfied . The immediate above relation is satisfied only if

$$\therefore y_0+\tfrac{1}{3}x_0^2=0 \qquad (3.30)$$

Hence the locus of (3.30) is thus obtained by

$$y+\tfrac{1}{3}x^2=0 \qquad (3.31)$$

Now using stable manifold theorem we have 1-dimensional stable manifold and 1-dimensional unstable manifold .
Hence clearly $y=-\frac{1}{3}x^2$ gives us the stable manifold W^s . On the other hand for obtaining the unstable manifold compute $\lim_{t\to-\infty}[(y_0+\frac{1}{3}x_0^2)e^t-\frac{1}{3}x_0^2e^{-2t}]=0$ and $\lim_{t\to-\infty}x_0e^{-t}=0$.
The above two relations are satisfied if

$$\therefore x_0=0 \qquad (3.32)$$

Hence the locus of the equation (3.32) is the y-axis . And this is the required unstable manifold .

The stable manifold can be expressed as

$$W^s(0)=\{(x,y)\in\mathbb{R}^2:y=-\tfrac{1}{3}x^2\}$$

The stable generalized eigenspace $E^s=\{(x,y)\in\mathbb{R}^2:y=0\}$ is clearly tangent to the stable manifold $W^s(0)$ at $(0,0)$.

3.7 Locally stable and unstable manifolds

By the word local we exactly mean what we want can expect thatis we are actually bound the flow of a smooth vector field in a open neighbourhood V of an equilibrium point . Let Γ be periodic orbit (hyperbolic).Then construct
$S(\Gamma) = \{x \in V : d(\phi(t, x), \Gamma) \to 0 \ \ as \ \ t \to \infty$ and $\phi(t, x) \subset V$ for $t \geq 0\}$
And also
$U(\Gamma) = \{x \in V : d(\phi(t, x), \Gamma) \to 0 \ \ as \ \ t \to -\infty$ and $\phi(t, x) \subset V$ for $t \leq 0\}$
$S(\Gamma)$ and $U(\Gamma)$ are said to be stable and unstable manifolds (but locally) of Γ . Clearly these are invariant manifolds as the flow does not leave the open neighborhood V of the fixed point .
That is we might surely have ,
$\phi(t, S(\Gamma)) \subset S(\Gamma)$ and $\phi(t, U(\Gamma)) \subset U(\Gamma)$. To extend these globally take the maximum atlas

$$W^s(\Gamma) = \bigcup_{t \geq 0} \phi(t, S(\Gamma)) \tag{3.33}$$

Again,

$$W^u(\Gamma) = \bigcup_{t \leq 0} \phi(t, U(\Gamma)) \tag{3.34}$$

3.8 Hamiltonian systems

The study of Hamiltonian systems of a conservative field in a plane is basically very helpful to model the dynamical systems where there is no loss of energy that is conservative . It is served key tool for a observing the separate cycles and their stability which we will introduce later.

3.8.1 Definition

By analogy with the form of Hamiltonian canonical equations in classical mechanics a system

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

is called a Hamiltonian system if there exists a function $H(x, y)$ such that $P = \dfrac{\partial H}{\partial y}$ and $Q = -\dfrac{\partial H}{\partial x}$.
Then H is called the Hamiltonian function for the system . If $H(x, y)$ be the total energy then

$$H(x, y) = K(x, y) + V(x, y) \tag{3.35}$$

where $K(x, y)$ is the kinetic and $V(x, y)$ is the potential energy.

Example

Consider the equation of motion of undamped one dimensional harmonic oscillator,

$$\ddot{x} + kx = 0 \tag{3.36}$$

The given dynamical system can be re-written as the planar non-linear system as

$$\dot{x} = y$$
$$\dot{y} = -kx$$

Now consider the function $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}kx^2$
$\therefore \dfrac{\partial H}{\partial y} = y$ and $\dfrac{\partial H}{\partial x} = -kx$.
Hence there exists a function $H(x, y)$ such that $\dot{x} = \dfrac{\partial H}{\partial y}$ and $\dot{y} = -\dfrac{\partial H}{\partial x}$.
Hence the given system is a Hamiltonian system .

A necessary and sufficient condition for a system to be Hamiltonian

Let us consider the dynamical system in $\mathbb{R}^2$ as

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

Then the system is said to be Hamiltonian if and only if

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \tag{3.37}$$

Proof

The condition is necessary

Let the given system be a Hamiltonian system . Then there exists a function $H(x, y)$ such that

$$P = \dot{x} = \frac{\partial H}{\partial y} \tag{3.38}$$

$$Q = \dot{y} = -\frac{\partial H}{\partial x} \tag{3.39}$$

Therefore

$$\begin{aligned} &\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial H}{\partial x}\right) \\ &= \frac{\partial^2 H}{\partial x \partial y} - \frac{\partial^2 H}{\partial y \partial x} \\ &= 0 \end{aligned}$$

The condition is sufficient

Let the condition $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$ hold . Now we consider the differential equation

$$-\dot{y}dx + \dot{x}dy = 0 \tag{3.40}$$

$$\implies Mdx + Ndy = 0 \tag{3.41}$$

where $M = -\dot{y} = -Q$, $N = \dot{x} = P$

$$\begin{aligned} &\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \\ &= -\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \\ &= -\left(\frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x}\right) \\ &= 0 \end{aligned}$$

Therefore the equation (3.40) is exact and hence there exists a function $H(x, y)$ such that the $L.H.S$ of (3.40) can be expressed as $d[H(x, y)]$.

Therefore

$$Mdx + Ndy = d[H(x, y)] = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy$$

Hence comparing the both sides we obtain $M = -\frac{\partial H}{\partial x}$ and $N = \frac{\partial H}{\partial y}$

3.9 Homoclinic and heteroclinic orbits

Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x + x^2 \end{aligned}$$

To find the expression for the Hamiltonian consider a function H so that $\frac{\partial H}{\partial y} = y$ and $\frac{\partial H}{\partial x} = -x - x^2$. So the total differential

$$\begin{aligned} dH &= \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy \\ &= ydy + (-x - x^2)dx \end{aligned}$$

Thus

$$H(x, y) = \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + C_1 \tag{3.42}$$

The Hamiltonian has the solution curve

$$y^2 - x^2 - \tfrac{2}{3}x^3 = C \tag{3.43}$$

At this moment we need to look at the phase portrait carefully to trace out the stable and unstable manifolds .

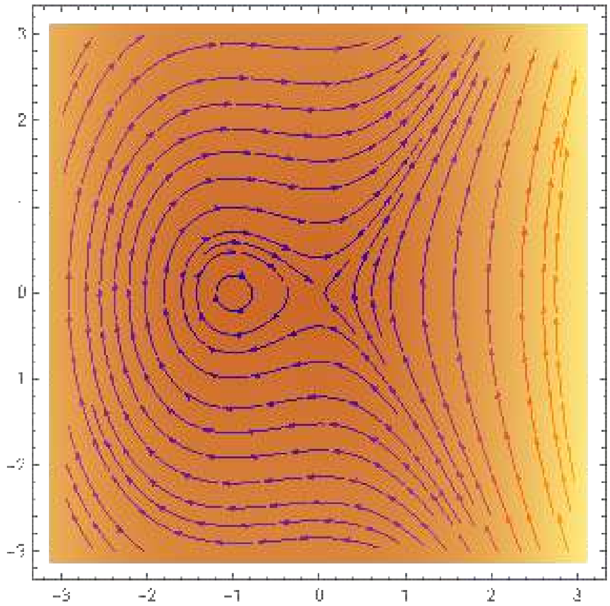

Figure 3.10: The homoclinic path

Now it is easier to rule out the stable and unstable manifolds by looking at the origin where the solution curve say Γ meets . The stability can also be justified by verifying the flow towards the origin or away from the origin . Hence clearly $\Gamma \subset W^s(0) \cap W^u(0)$. Observe that the equilibrium points of the given system are $(0,0)$ and $(-1,0)$ respectively. A homoclinic orbit together with the equilibrium points *i.e* the union $S_{Hom}^{cycle} := \Gamma \cup \{(0,0)\} \cup \{(-1,0)\}$ is called a ***Separatrix cycle*** .

Again take the system

$$\dot{x} = y$$
$$\dot{y} = -x + x^3$$

First to obtain the equilibrium points write $\dot{x} = 0$ and $\dot{y} = -x + x^3$
$\implies y = 0$ and $x = 0, \pm 1$. Hence the equilibrium points are $(0,0)$, $(-1,0)$,$(1,0)$.
Let $V(x)$ denote the potential function. Then

$$
\begin{aligned}
V'(x) = -\dot{y} = x - x^3 \\
V''(x) = 1 - 3x^2 \\
V''(0) > 0 \text{ (minimum)} \\
V''(-1) < 0 \text{ (maximum)} \\
V''(1) < 0 \text{ (maximum)}
\end{aligned}
$$

$\therefore$ The point $(0,0)$ is a center and other two points are saddle points respectively .
To verify this let us have a look on the phase portrait

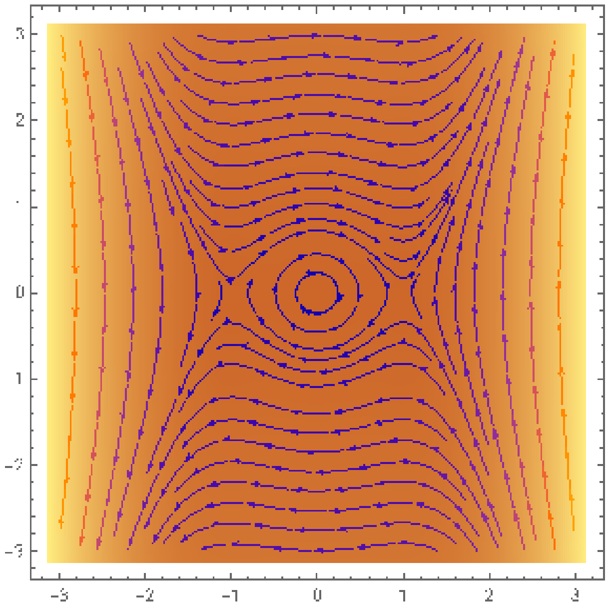

Figure 3.11: Heteroclinic path

Hence by the similar attempts as above we should have a Hamiltonian and from the symmetries of upper and lower half planes we have two heteroclinc orbits , say Ω_1 and Ω_2. Again the solution curve together with the equilibrium points is thus denoted as $S_{Het}^{Cycle} = \Omega_1 \cup \Omega_2 \cup \{(0,0)\} \cup \{(-1,0)\} \cup \{(1,0)\}$ is called a ***Compound Separatrix Cycle***.

3.10 Rotated Vector Fields

Consider the equations

$$
\begin{aligned}
\dot{x} &= P(x, y, \epsilon) \\
\dot{y} &= Q(x, y, \epsilon)
\end{aligned}
\tag{3.44}
$$

where P and Q are at least continuous functions of x and y and ϵ be the parameter satisfying the following conditions :

1. P and Q are Lipschtiz in any bounded region .

2. $\dfrac{\partial P}{\partial \epsilon}$ and $\dfrac{\partial Q}{\partial \epsilon}$ exist and are continuous .

3. The vector field $\chi(\epsilon)$ defined by the system (3.44) has only isolated singular points .
   Let ϵ varies in $[0, T]$ where T is any time interval ,singular points of $\chi(\epsilon)$ remain unchanged and at all regular points we have
   $det\begin{pmatrix} P & Q \\ \dfrac{\partial P}{\partial \epsilon} & \dfrac{\partial Q}{\partial \epsilon} \end{pmatrix} > 0$

Again let there exists a positive function $k(x, y)$ such that for all $(x, y), T > 0$, the smallest positive number for which

$$\begin{aligned} P(x, y, \epsilon + T) &= -kP(x, y, \epsilon) \\ Q(x, y, \epsilon + T) &= -kP(x, y, \epsilon) \end{aligned} \tag{3.45}$$

Then we say $\chi(\epsilon)$ forms a *complete family of rotated vector fields*

3.10.1 Theorem

Suppose the system (3.44) forms a complete family of rotated field with respect to ϵ and Γ_0 be a separatrix cycle of the vector field $\chi(\epsilon_0)$ passing through a saddle point . Now if at that saddle point

$$\frac{\partial P(x, y, \epsilon_0)}{\partial x} + \frac{\partial Q(x, y, \epsilon_0)}{\partial y} < 0(\text{ or } > 0) \tag{3.46}$$

then when ϵ varies from ϵ_0 in a suitable sense then , a unique stable (or unstable) limit cycle of $\chi(\epsilon)$ will be generated close to the inside of Γ_0 and when ϵ varies in opposite sense , Γ_0 disappears and there exists no limit cycle in its neighborhood.

3.11 Stability of Van Der Pol equation using energy balance method

Consider the equation

$$\ddot{x} + \epsilon h(x, \dot{x}) + g(x) = 0 \tag{3.47}$$

ϵ being a small positive quantity . Writing down the above equation as a first order planar system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\epsilon h(x, y) - g(x) \end{aligned} \tag{3.48}$$

For equilibrium points

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \end{aligned} \tag{3.49}$$

Hence

$$\begin{aligned} y &= 0 \\ \epsilon h(x,y) + g(x) &= 0 \\ \epsilon h(x,0) + g(x) &= 0 \end{aligned} \tag{3.50}$$

Let us assume $g(0) = 0$ where $(0,0)$ is the only equilibrium point . This gives

$$\begin{aligned} &\epsilon h(x,0) + g(x) = 0 \\ \Longrightarrow\ &\epsilon h(0,0) + g(0) = 0 \\ \Longrightarrow\ &h(0,0) = 0 \end{aligned}$$

Now take the potential energy $V(x) = \int g(x)dx$ and the kinetic energy $T = \frac{1}{2}\dot{x}^2$. The total energy

$$\begin{aligned} E = T + V &= \tfrac{1}{2}\dot{x}^2 + \int g(x)dx \\ \frac{dE}{dt} &= [\ddot{x} + g(x)]\dot{x} \end{aligned}$$

Again from the equation (3.47)

$$\ddot{x} + g(x) = -\epsilon h(x,\dot{x}) \tag{3.51}$$

Hence we obtain

$$\frac{dE}{dt} = -\epsilon h(x,\dot{x})y \tag{3.52}$$

Now for a time interval $[0,\tau]$, integrating (3.52)

$$E(\tau) - E(0) = \int_0^\tau -\epsilon h(x,\dot{x})ydt$$

In particular for $\tau = 2\pi$ we call the function

$$g(a) = E(2\pi) - E(0) = \int_0^{2\pi} -\epsilon h(x,\dot{x})ydt \tag{3.53}$$

3.11.1 Establishment of energy balance equation and stability of limit cycle for the Van Der Pol equation

Consider the Van Der Pol equation in the form as

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{3.54}$$

ϵ being small .

Solution

To study the stability of limit cycle of the Van Der Pol equation here ,

$$h(x, y) = (x^2 - 1)\dot{x} = (x^2 - 1)y$$

For the approximate amplitude assume $x \approx a\cos t$, $y \approx a\sin t$ and $\tau \approx 2\pi$
Hence we obtain

$$g(a) = \epsilon a \int_0^{2\pi} h(a\cos t, -a\sin t).sintdt = 0 \tag{3.55}$$

Equation (3.55) is called the ***energy balance equation*** .
Now compute

$$\begin{aligned}
g(a) &= \epsilon a \int_0^{2\pi} h(a\cos t, -a\sin t).\sin t dt \\
&= \epsilon a \int_0^{2\pi} (a^2\cos^2 t - 1).(-a\sin t)\sin t dt \\
&= -\epsilon a^2 \int_0^{2\pi} (a^2\cos^2 t - 1).\sin^2 t dt \\
&= -\epsilon a^2 \{\int_0^{2\pi} a^2\cos^2 t\sin^2 t dt - \int_0^{2\pi} \sin^2 t dt\} \\
&= -\tfrac{1}{2}\epsilon a^2\{\tfrac{a^2}{4} \int_0^{2\pi} (1 - \cos 4t)dt - \int_0^{2\pi} (1 - \cos 2t)\}dt \\
&= -\tfrac{1}{2}\epsilon a^2.2\pi(\tfrac{a^2}{4} - 1) \\
&= -\epsilon\pi a^2(\tfrac{a^2}{4} - 1)
\end{aligned}$$

Therefore ,

$$g'(a) = -\epsilon\pi(a^3 - 2a) \tag{3.56}$$

Hence

$$\begin{aligned}
g(a) &= 0 \\
\implies \tfrac{a^2}{4} - 1 &= 0 \\
\implies a &= 2
\end{aligned}$$

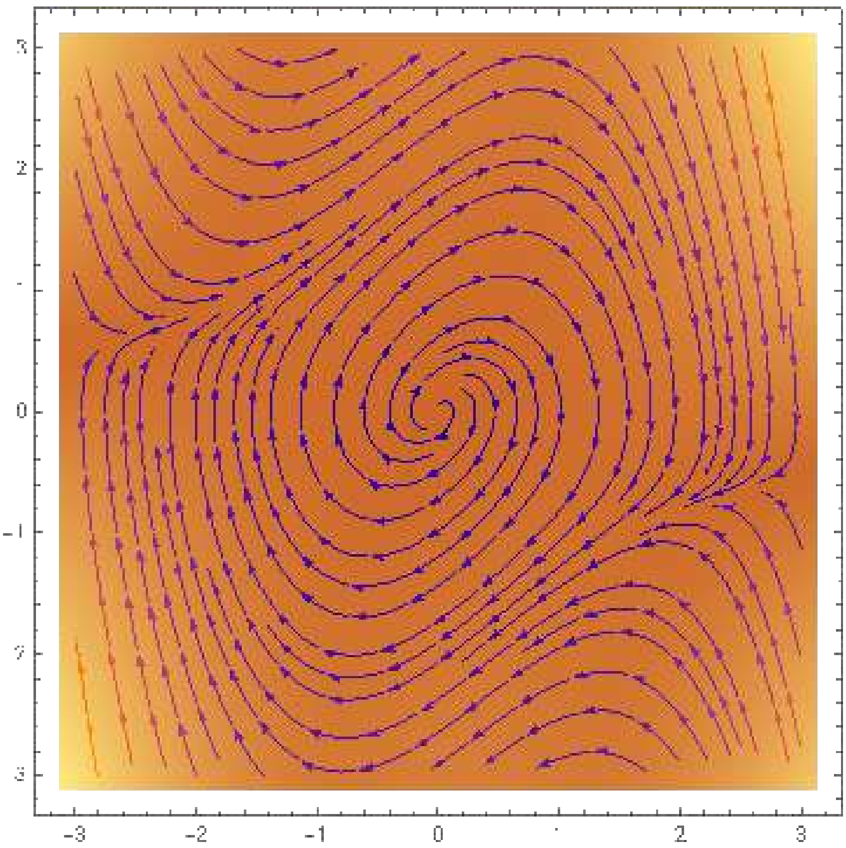

Figure 3.12: limit cycle for the Van Der Pol equation

Now putting $a = 2$ in (3.56) we get

$$g'(a) = -\epsilon\pi(8-4)$$
$$= -2\pi\epsilon$$

So the limit cycles are stable or unstable according as in either case for $\epsilon > 0$ or $\epsilon < 0$.

Chapter 4

Existence Of Limit Cycle

4.1 Introduction

In this chapter we so far became a little bit aware of limit cycles how it looks or its stability . But one should certainly ask the existence of a limit cycle and its numbers . At this point we mention that the problem of finding the existence or the total or maximum number of limit cycles of the planner autonomous system given by ,

$$\begin{aligned} \dot{x} &= X(x, y) \\ \dot{y} &= Y(x, y) \end{aligned} \tag{4.1}$$

was a challenging problem for decades . Especially the total number of limit cycles . It is a century old problem that was first coined and introduced by the renowned mathematician Hilbert . Thus this problem is known as Hilbert's 16^{th} problem .
Here , in this chapter we are mainly focusing on the on some basic ideas behind the theory of existence of limit cycles of a system like as given in (4.1) and mainly work on a special class of non-linear differential equations known as the Liénard system .

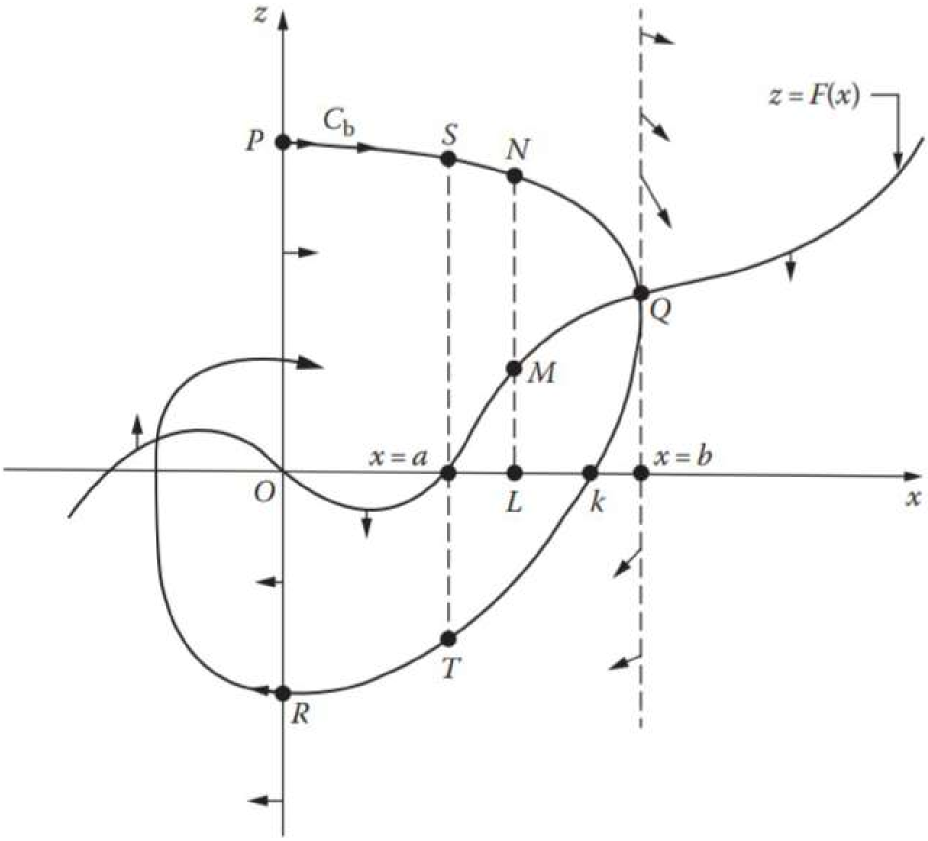

Figure 4.1: The path of the differential equation

4.2 Classical Liènard system

Now consider the equation given by ,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{4.2}$$

First assume that $f(x)$ and $g(x)$ satisfy few conditions given below-

1. $f(x)$ and $g(x)$ are smooth .
2. $g(x)$ being an odd function so that $g(x) > 0$ if $x > 0$ and $f(x)$ is an even function.
3. The odd function F(x) $= \int_0^x f(\xi)\, d\xi$ has exactly one positive zero at $x = a$ and is negative whenever $0 < x < a$, is positive and non-decreasing for $x > a$.
4. $F(x) \to \infty$ as $x \to \infty$

We shall prove that the system (4.2) has a unique closed path surrounding the origin in the phase plane .

Consider the equivalent system to (4.2) as

$$\dot{x} = y \tag{4.3}$$

$$\dot{y} = -g(x) - f(x)y \tag{4.4}$$

By the first condition, using Picard's theorem on the existence and uniqueness of the solution of (4.3) and (4.4) holds good .
Again $g(0) = 0$ and $g(x) \neq 0$ whenever $x \neq 0$.
So clearly the origin is the only critical point here and also it is a well known fact that any closed path must surround the origin . Now observe that

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} = \frac{d}{dt}\left[\frac{dx}{dt} + \int_0^x f(x)dx\right] \tag{4.5}$$

We introduce a new variable here , say z so that $z = y + F(x)$.
Also

$$\frac{dz}{dt} = \frac{dy}{dt} + f(x)\frac{dx}{dt} \tag{4.6}$$

Then

$$\frac{dz}{dt} = \dot{y} + f(x)y. \tag{4.7}$$

So ,

$$\frac{dz}{dt} = -g(x) - f(x)y + f(x)y \tag{4.8}$$

Hence

$$\frac{dz}{dt} = -g(x) \tag{4.9}$$

Hence the system one is equivalently be expressed as

$$\dot{x} = z - F(x) \tag{4.10}$$

$$\dot{z} = -g(x). \tag{4.11}$$

Again we note that one fact that since all these functions are smooth , the existence and uniqueness of the solution again holds good for the newly obtained system . Similarly any closed path must surround the origin .

We define a map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ by ,

$$\rho(x, y) = (z - F(x), g(x)) \ , \forall (x, y) \in \mathbb{R}^2 \tag{4.12}$$

Clearly ρ is a diffeomorphism (categorically speaking) . So , there is obviously a one to one correspondence $(x, y) \mapsto (x, z)$ between the points of two the planes that signifies the correspondence between the paths (closed paths corresponds to the close paths) and the configurations of the paths in two planes are qualitively similar .

Hence the differential equation of the paths is thus obtained by -

$$\frac{dz}{dx} = -\frac{g(x)}{z - F(x)} \tag{4.13}$$

Again note that we analyse here the paths in (4.13) but not the corresponding paths in the phase plane as we do mostly . There are few reasons why are we even doing so .

1. Firstly , as a we say that since $g(x)$ and $F(x)$ are both odd functions, the equations (4.10) , (4.11) and (4.13) remain unchanged when we replace x and z by $-x$ and $-z$.
   This means that any curve becomes symmetric to a path about the origin . Thus to precised if we know the nature of the curve in the right half-plane denoted by $\mathbb{H}^2_+$ as $x > 0$, we can easily conclude what is going to be in the left-half plane , denoted as $\mathbb{H}^2_-$ as $x < 0$.

2. Secondly , From the equation (4.13) observe that the paths become horizontal only as they cross the z-axis and they again become vertical when the paths cross the curve

   $$z = F(x) \tag{4.14}$$

   Now follow the nullclines in the neighborhood of the curve $z = F(x)$ whenever one sees the curve from the segment (portion) of the curve , the paths show a tendency to approach the curve .

   Also whenever one sees it from below the curve the paths show a tendency to get repelled (roughly speaking) from the curve .
   Also note that the curve $z = F(x)$, the z-axis and the vertical line these can only be crossed when the arrows in the given illustration moves so in the pointed manner as given in the illustration in figure 4.1.

Now consider this solution of (4.2) defines a path say C passing through the point Q be such that the point Q correspond to the parameter t whenever $t = 0$. Now as t is increasing the curve C moves down , still moves continuously until it reaches the point R .
Then it is the turn t to be decreased and it gives rise to the path C from the left side of the z-axis , here t takes the negative values unlike the right side where t took only the positive values . The points on C rises to the left until it crosses the z-axis at the point p .
Now this entire phenomenon is occurring with respect to the points $(x(t), y(t)) \quad \forall\, t \in \mathbb{R}$
Here , we name an abscissa say b of Q and denote the C as C_b which makes sense . Due to the nice symmetry of both upper and lower half plane C_b is continued in both the

planes and evidently we are done if $OP = OQ$. More precisely,we will see a closed path if and only if

$$OP = OQ \tag{4.15}$$

holds .
And further to show the uniqueness it will be sufficient to show that ∃ b∈ ℝ uniquely so that (4.15) holds .

Now consider the function ,

$$G(x) = \int_0^x g(\xi)\, d\xi \tag{4.16}$$

Again we define a smooth function -

$$E(x, z) = \tfrac{1}{2}z^2 + G(x) \tag{4.17}$$

The equation (4.17) reduces to just $\frac{1}{2}z^2$ when $x = 0$ that is when it is on the z-axis .

Along any path we have

$$\frac{dE}{dt} = g(x)\frac{dx}{dt} + z\frac{dz}{dt} \tag{4.18}$$

$$= -[z - F(x)]\frac{dz}{dt} + z\frac{dz}{dt} \tag{4.19}$$

$$= F(x)\frac{dz}{dt} \tag{4.20}$$

So

$$dE = Fdz \tag{4.21}$$

Now compute the line Integral of Fdz along the path C_b from P to R , we obtain

$$I(b) = \int_{PR} Fdz = \int_{PR} dE = E_R - E_P = \tfrac{1}{2}(OR^2 - OP^2) \tag{4.22}$$

So it suffices to show that there is a unique b such that $I(b) = 0$.
Now if $b \leq a$, then F and dz are negative, so $I(b) > 0$ and C_b cannot be closed.
Suppose now that $b > a$ as given in the figure . We now separate $I(b)$ into two parts respectively for our convenience

$$I_1(b) = \int_{PS} Fdz + \int_{TR} Fdz \tag{4.23}$$

And

$$I_2(b) = \int_{ST} Fdz \tag{4.24}$$

So that

$$I(b) = I_1(b) + I_2(b) \tag{4.25}$$

Since F and dz are negative as C_b is traversed from P to S and from T to R , it implies that $I_1(b) > 0$. Again , if we move from S to T along C_b we see $F > 0$ and $dz < 0$ implies that $I_2 < 0$. Mark the goal first that we are to show that I_b is a decreasing function of b by considering $I_1(b)$ and $I_2(b)$. Note that from equation (4.13) we must have

$$Fdz = F\frac{dz}{dx}dx = \tfrac{-g(x)F(x)}{z-F(x)}dx \tag{4.26}$$

the implications of increasing be used to raise the arc PS and to decrease the arc TR , which decreases the magnitude of $\frac{-g(x)F(x)}{z-F(x)}$ for a given X between 0 and a . Since the limits of integration for $I_1(b)$ are fixed , the result is a decrease in $I_1(b)$.
Further since $F(x)$ is positive and it is non-decreasing to the right of a , we observe that an increment in b gives rise to an increment in the positive number $-I_1(b)$ and hence to a decrement in $I_2(b)$.
In this way we have $I_2(b) \to -\infty$ as $b \to \infty$. Now if L in the above figure is being fixed and K is to the right of L,then we have

$$I_2(b) = \int_{ST} Fdz < \int_{NK} fdz \leq -(LM).(LN) \tag{4.27}$$

and since $LN \to \infty$ as $b \to \infty$, we have $I_2(b) \to \infty$.
Note that $I(b)$ is a decreasing continuous function of b for $b \geq a$, $I(b) > 0$ and $I(b) - \infty$ as $b \to \infty$. It follows that , $I(b) = 0$ for one and only one $b = b_0$. So there is one and only closed path C_{b_0} .
Finally , we observe that $OR > OP$ for $b < b_0$; and from this and the symmetry we conclude that paths inside C_{b_0} spiral out C_{b_0} . Similarly the fact that $OR < OP$ for $b > b_0$ implies that the paths outside C_{b_0} spiral into C_{b_0} .

Applications

4.2.1 Revised version of the Liènard theorem

If the conditions from (1) to (4) as given above are satisfied,the equation (4.2) has a unique stable limit cycle .

We have already seen the stability of the limit cycles for the Van Der Pol equation in the previous chapter-3 .

Verification the Liènard theorem for the Van Der Pol equation

For the Van Der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{4.28}$$

Let $f(x) = \epsilon(x^2 - 1)$ and $g(x) = x$. Then clearly

1. $f(x)$ and $g(x)$ are smooth .
2. $g(x) = x$ and $g(-x) = -x = -g(x)$
3. $g(x) > 0$ for $x > 0$
4. $f(-x) = f(x)$ that is $f(x)$ is an even function .
5. Lastly $F(x) = \int_0^x f(\xi)d\xi$ is an odd function since for

$$\begin{aligned} F(x) &= \int_0^x \epsilon(\xi^2 - 1)d\xi \\ &= \tfrac{1}{3}\epsilon x^3 - \epsilon x \end{aligned}$$

Now put

$$\begin{aligned} F(x) &= -\tfrac{1}{3}\epsilon x^3 + \epsilon x \\ &= -F(x) \end{aligned}$$

has positive zero at $x = \sqrt{3}$, is negative for $0 < x < \sqrt{3}$, is positive and non-decreasing for $x > \sqrt{3}$. Hence the given system has only one closed path. The closed path is a limit cycle .

4.3 Poincaré-Bendixson Theorem and its applications

Let Ω be a closed and bounded region that contains no equilibrium point of the planar system (4.1) such that for some positive half-trajectory Γ of the system lies entirely within Ω . Then either Γ itself a closed path or it approaches asymptotically to a closed path as either $t \to \infty$ or $t \to -\infty$

4.3.1 Example

Consider the system

$$\begin{aligned} \dot{x} &= x(1 - x^2 - y^2) - y \\ \dot{y} &= y(1 - x^2 - y^2) - x \end{aligned} \tag{4.29}$$

To obtain the equilibrium points equalize the equations to zero i.e.

$$\begin{aligned} \dot{x} = x(1 - x^2 - y^2) - y = 0 \\ \dot{y} = y(1 - x^2 - y^2) - x = 0 \end{aligned}$$

Note that origin is the only fixed point here . As earlier we transformed the system into polar co-ordinates and obtained the equations

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= -1 \end{aligned} \qquad (4.30)$$

A particular solution thus obtained by

$$r = 1$$
$$\theta = -t$$

And this clearly corresponds to the limit cycle
$x = \cos t$, $y = -\sin t$.
Observe that $\dot{r} > 0$ for $0 < r < 1$ that clearly depicts the fact that phase paths approach the limit cycle $r = 1$ from inside (ω-limit set) and again note that $\dot{r} < 0$ for $r > 1$ and this implies that the phase paths approach the limit cycle $r = 1$ from outside (ω-limit set) . And in order to say it collectively , $r = 1$ is stable limit cycle . The equation for $\dot{\theta}$ shows that the points on the phase path move spirally in the clockwise direction around the limit cycle . Thhe phase portrait is shown in the figure 4.2 .

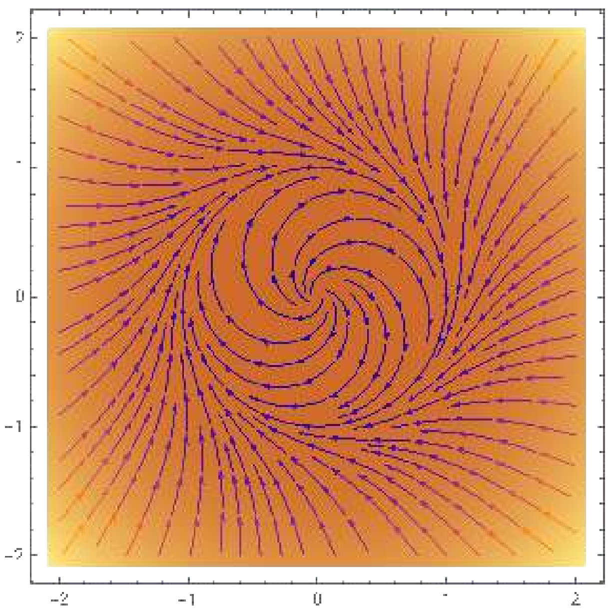

Figure 4.2: Stable limit cycle

Again consider the system

$$\begin{aligned} \dot{x} &= -x(1-x^2-y^2)+y \\ \dot{y} &= -y(1-x^2-y^2)+x \end{aligned} \qquad (4.31)$$

The phase portrait for the system which is directed negatively. It is shown in the figure 4.3.

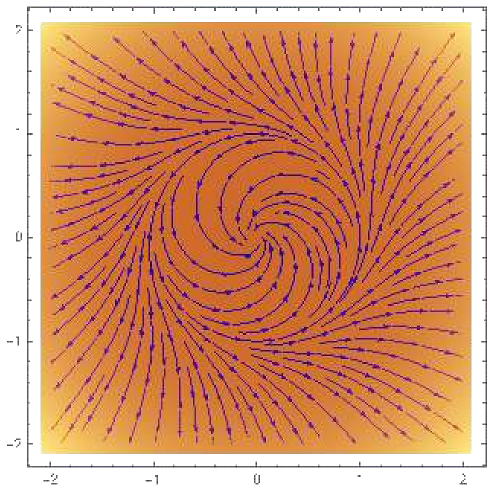

Figure 4.3: Unstable limit cycle

4.3.2 Morphology of neuron

The morphology of neurons is a technique of studying of neuronal structure, is often used to identify and classify neurons .

The neuron is the basic unit of the nervous system. It is an excitable cell that has been

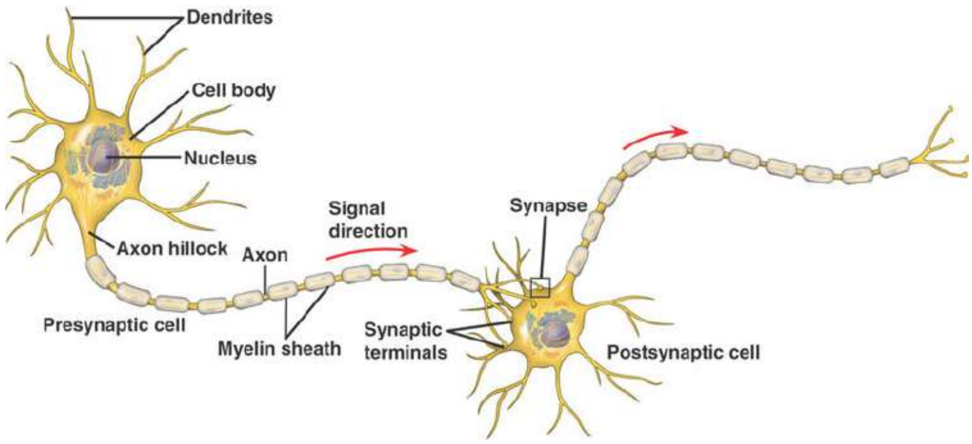

Figure 4.4: A typical morphology of a neuron

adapted morphologically for the sake simplicity and better comprehension . By staining neurons we mean to study the structure, pathology, and function of neural tissue in the brain and spinal cord .

Fitzhugh-Nagumo model

This particular mathematical modelling deals with the excitability of single neuron . Very briefly and roughly speaking a neuron can produce the electric impulses in response to any stimulation . This mathematical model is given by the pair of equations

$$\begin{aligned} \dot{x} &= x - \tfrac{1}{3}x^3 - y + \mathcal{I} \\ \dot{y} &= \epsilon(\beta x - y - \alpha) \end{aligned} \tag{4.32}$$

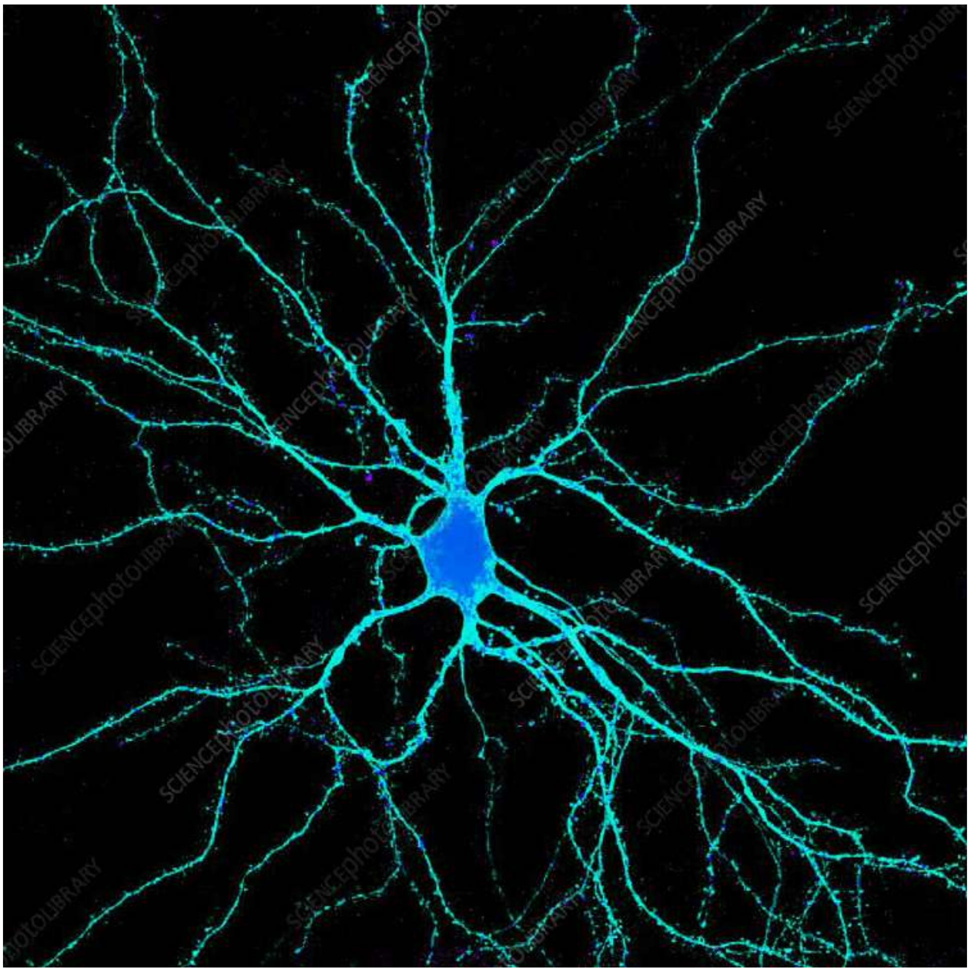

Figure 4.5: Stained single neuron

where $\epsilon << 1$, $0 < \alpha < 1$ and $\beta > 1$. Also let $x(t)$ and $y(t)$ denote the membrane potential of the neurone and the excitability of the neuron respectively. The parameters ϵ , α and β represent fixed physical and biological properties of the cell and $\mathcal{I}$ current being put into the neuron. in order to obtain a simplified analytical standard result we need to fix our parameters and the input current first by choosing

$$\epsilon = 0.08$$
$$\alpha = 0.3$$
$$\beta = 6$$
$$\mathcal{I} = 1$$

The equilibrium points are given by

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \end{aligned} \tag{4.33}$$

This implies

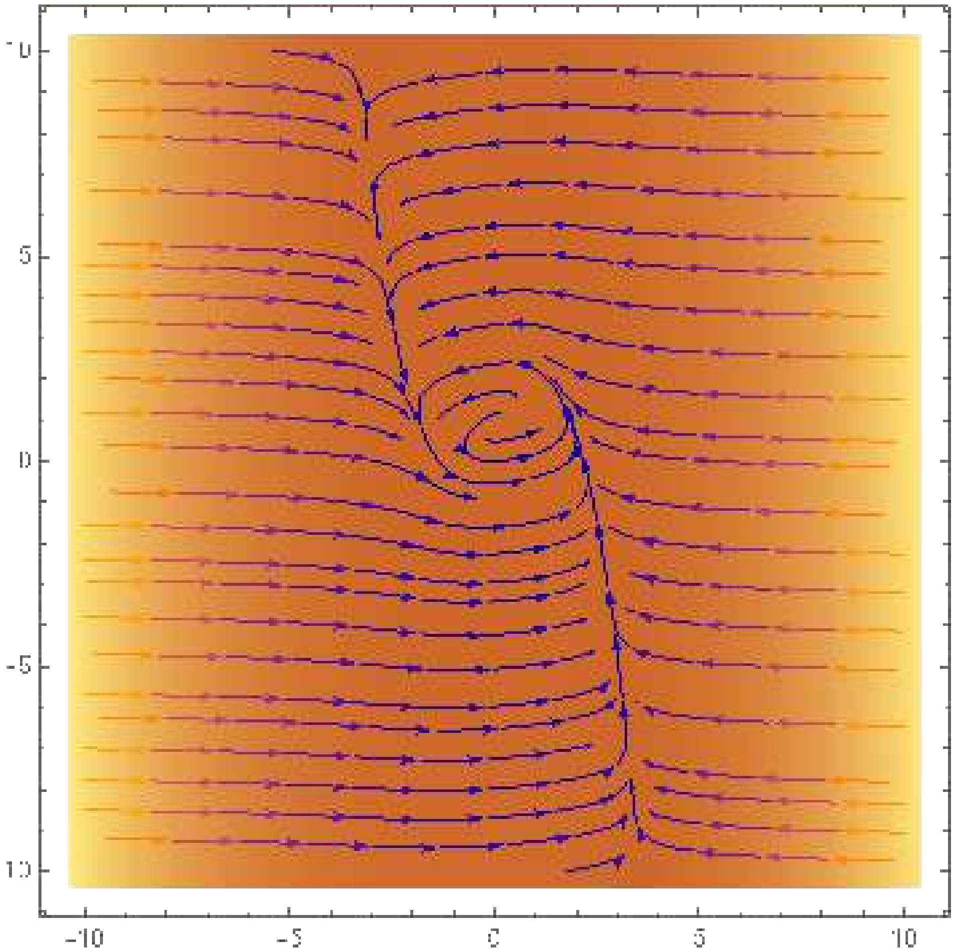

Figure 4.6: Vector field plot of Fitzhugh-Nagumo model

$$\begin{array}{c} x - \frac{1}{3}x^3 - y + \mathcal{I} = 0 \\ \epsilon(\beta x - y - \alpha) = 0 \end{array} \tag{4.34}$$

Putting the fixed standard values of the parameters and the input current

$$x - \frac{1}{3}x^3 - y + 1 = 0 \tag{4.35}$$

$$6x - y = -0.3 \tag{4.36}$$

Now from equation (4.36) we put $y = 6x + 0.3$ in equation (4.35) and hence we get

$$x - \frac{1}{3}x^3 - 6x - 0.3 + 1 = 0 \tag{4.37}$$

Therefore ,

$$x^3 + 15x - 2.1 = 0 \tag{4.38}$$

It will be easier to solve using Cardan's method and so let $x = u + v$. This implies that

$$x^3 = u^3 + v^3 + 3uvx \tag{4.39}$$

.
That is

$$x^3 - 3uvx - (u^3 + v^3) = 0 \tag{4.40}$$

Now comparing (4.38) and (4.40) we obtain

$$3uv = -15$$

$$uv = -5 \tag{4.41}$$

and

$$u^3 + v^3 = 2.1 \tag{4.42}$$

Now from (4.41) putting the value of $v = -\frac{5}{u}$ in equation (4.42)

$$u^3 - \tfrac{125}{u^3} = 2.1$$

Let $t = u^3$ then we have

$$t - \tfrac{125}{t} = 2.1$$
$$t^2 - 2.1t - 125 = 0$$

Solving the equation we obtain

$$t = 12.279$$
$$u^3 = 12.279$$
$$u = 2.30709$$

Hence $v = -2.16728$. So
$x = u + v = 2.30703 - 2.16728 = 0.14$ and this implies that $y = 6 \times 0.14 + 0.3 = 1.14$.
Hence the fixed point is $(0.14, 1.14)$.

To classify the fixed point write down the Jacobian
$Jf(x, y) = \dfrac{\partial(\dot{x}, \dot{y})}{\partial(x, y)} = \begin{pmatrix} 1 - x^2 & -1 \\ \beta\epsilon & -\epsilon \end{pmatrix}$. Putting the chosen values of the parameters we

obtain the Jacobian as $\begin{pmatrix} 1-x^2 & -1 \\ 0.48 & -0.08 \end{pmatrix}$

At the fixed point $(0.14, 1.14)$,

$$\left.\frac{\partial(\dot{x}, \dot{y})}{\partial(x, y)}\right|_{(0.14,1.14)} = \begin{pmatrix} 0.9804 & -1 \\ 0.48 & -0.08 \end{pmatrix}$$

. Now the characteristic equation is obtained by

$$\lambda^2 - 0.9004\lambda + 0.40158 = 0 \tag{4.43}$$

Hence solving (4.43) , $\lambda = 0.4502 \pm 0.445965i$. So the fixed point is an unstable spiral and precisely a source(see figure-4.7) of the approximated linear system of (4.34) . Since the real part of the eigen values are non zero , by Hartman-Großman theorem,it can be concluded that the phase trajectory of linear system will be locally topologically conjugate with the actual nonlinear system .

To show the existence of a limit cycle we construct a closed positively invariant region

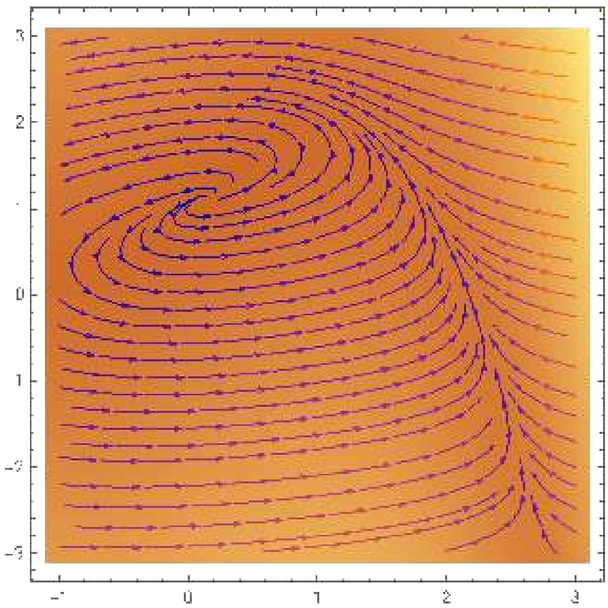

Figure 4.7: phase trajectory of the approximated linear system

in the plane . Since the equilibrium point $(0.14, 1.14)$ is a source we can find a small open neighbourhood N of the equilibrium point so that on ∂N , the vector field points away from N . Let L denote the line defined by $y' = 0$. Let A be the point where $x = 20$ intersects L and let B denote the point where $x = -20$ intersects L . Let K denote a closed rectangle whose diagonal is connecting the points A and B . We can surely consider N is sufficiently small to be contained within the interior of K .

We will show that $K \setminus N$ is positively invariant. The line L divides the plane into two regions where $y' > 0$ and $y' < 0$. Since the upper boundary of K is in the region $y' < 0$ and the lower boundary is in the region $y' > 0$, N points towards the interior of K on the upper or lower parts binding it . As we divided the plane into two regions horizontally . Similarly we divide the plane vertically now in two regions $x' > 0$ and $x' < 0$. Thus ,

the $x' < 0$ on the right edge of K and $x' > 0$ on the left of the region K . So the open neighbourhood N points towards the interior of $K \setminus O$. So , $K \setminus O$ is a positively invariant set .
Again , $K \setminus O$ is closed and so the for any trajectory Γ , the ω-limit set $\omega(\Gamma) \subset K \setminus O$. This is enough to say that $\omega(\Gamma)$ is bounded and does not contain any equilibrium point . Hence by Poincaré-Bendixson theorem , there exists a closed orbit . And in particular we see that this is a closed orbit .

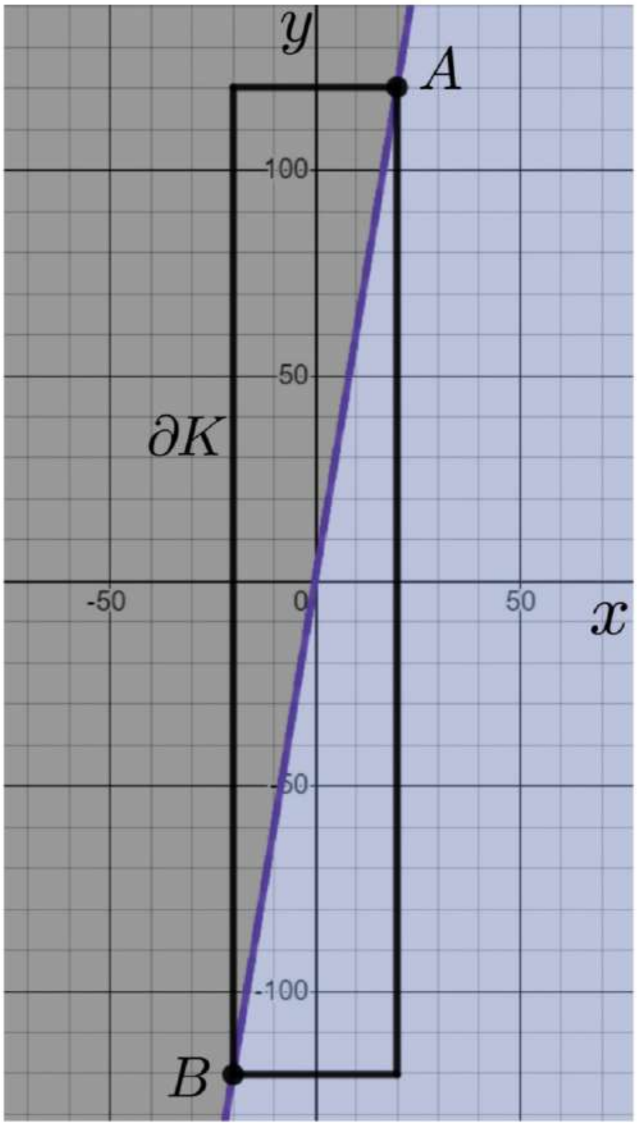

Figure 4.8: Plotting of the graph

Chapter 5

Non-Existence Of Limit Cycle

5.1 Introduction

In previous chapters we discussed on limit sets , criteria for a closed path to be a limit cycles or when it exists for a non-linear dynamical system . We also dealt with few examples regarding the existence of limit cycles which was supposed be a difficult task for a mathematician since one has to restrict the dynamical systems in a certain boundaries and we impose too many conditions to obtain a closed path or trajectory . Not only that we also had to show whether it is exactly a isolated closed trajectory. And this makes it little bit difficult to handle .
But now as this chapter is concerned we will be noticing saying something about the non-existence is easier than that of existence of limit cycles for a dynamical system.

Cauchy-Kovalevskya problem (ODE)

5.2 Real analyticity

A function f defined on some open subset U of $\mathbb{R}$ is said to be real analytic at any point $x_0 \in U$ if it is an infinitely differentiable function so that the Taylor series

$$\Psi(x) = \sum_{n=0}^{\infty} \tfrac{f^n(x_0)}{n!}(x - x_0)^n \tag{5.1}$$

converges to $f(x)$ for any x in some neighbourhood of x_0.
Also f is said to be real analytic on an open subset U of $\mathbb{R}$ if and only if for any compact set $K \subset U$ there exist constants M and $r(> 0)$ such that

$$\forall\, x \in K, |f^n(x)| \leq M \tfrac{n!}{r^n} \tag{5.2}$$

5.2.1 Theorem

Suppose $h > 0$ and $f : (y_0 - h, y_0 + h) \to \mathbb{R}$ be real analytic and $y(t)$ be the unique solution to the ODE

$$\dot{y} = f(y(t)) \;\; with \;\; y(0) = y_0 \in \mathbb{R} \tag{5.3}$$

on some neighborhood $(-y_0, y_0)$ of *zero* , with $y((-y_0, y_0)) \subset (y_0 - h, y_0 + h)$. Then y is also real analytic on $(-y_0, y_0)$

5.3 The dynamical evolution and controlled dynamics

We consider the ODE

$$\mathbf{D} : \dot{y}(t) = f(y(t)) \;\; , y(0) = y_0, t > 0 \tag{5.4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and $y_0 \in R^n$. The curve $\gamma : [0, \infty] \to \mathbb{R}^n$ is interpreted as a dynamical evolution of a state of a system $\mathbf{D}$, denoted as $\mathcal{S}(\mathbf{D})$.
Let f depend on some parameters say t and A being a set of such parameters . Consider the dynamics as

$$\dot{y}(t) = f(y(t), a) \;\; , y(0) = y_0, t > 0. \tag{5.5}$$

Now define a map $\alpha : [0, \infty) \to A$ by

$$\alpha(t) = \begin{cases} a_1, & \text{for } 0 \le t \le t_1 \\ a_2, & \text{for } t_1 < t \le t_2 \\ a_3, & \text{for } t_2 < t \le t_3 \\ a_4, & \text{for } t > t_3 \end{cases}$$

Now impose this function to the dynamics

$$\dot{y}(t) = f(y(t), \alpha(t)) \;\; , y(0) = y_0, t > 0. \tag{5.6}$$

Here α being measurable function *i.e* for any open set $\mathcal{V}$ in A , $\alpha^{-1}(\mathcal{V}) \in \mathcal{M}_c$ where $\mathcal{M}_c$ being any σ-algebra of a measure space . Also $\alpha(t)$ is of the form

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \\ \dots \\ \alpha_n(t) \end{pmatrix}$$

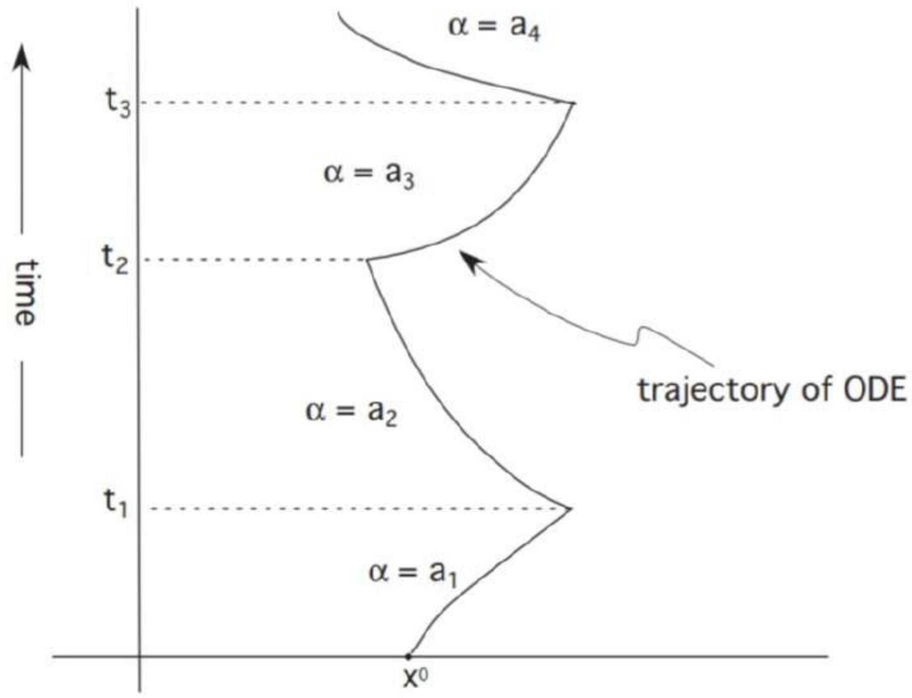

Figure 5.1: The dynamical Evolution

5.3.1 Input-output map

Consider the pair of C^∞ vector fields $(\chi_\mathbf{0}, \chi_\mathbf{1})$ on a finite dimensional C^∞ σ-compact manifold $(dim\ M \geq 3)$. Let $m \in M$ and $T > 0$ and consider the quadruple $(\chi_\mathbf{0}, \chi_\mathbf{1}, m, T)$. Now define a smooth $E_{m,T} : U(m,T) \to M$, where $U(m,T)$ is a set of all functions $u \in L^\infty([0,T]), \mathbb{R})$ such that the solution of the *Cauchy problem* :

$$\frac{dy}{dt}(t) = \chi_\mathbf{0}(\mathbf{y}(\mathbf{t})) + u(t)\chi_\mathbf{1}(\mathbf{y}(\mathbf{t}))\ ,\ y(0) = m \tag{5.7}$$

is defined on $[0,T]$. Then $E_{m,T}(u) = y(T) \in M$

5.3.2 Singular trajectory

Let M be a C^∞ σ-compact manifold of $dim\ M \geq 3$. Now consider a single input control system

$$\frac{dy}{dt}(t) = \chi_\mathbf{0}(\mathbf{y}(\mathbf{t})) + u(t)\chi_\mathbf{1}(\mathbf{y}(\mathbf{t})) \tag{5.8}$$

on M with the smooth vector fields $\chi_\mathbf{0}$ and $\chi_\mathbf{1}$ and let the set of admissible controls $\mathcal{U}$ is the set of bounded measurable mappings $y : [0,T] \to \mathbb{R}$. Then a singular trajectories and output corresponding to a control such that the differential of the input output mapping is not of maximal rank.[1]

5.4 Criteria for non-existence of limit cycles of a planar system

5.4.1 Bendixson negative criterion

Consider the system

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned} \tag{5.9}$$

Let Ω be a simply connected region . Then there are no closed paths in Ω of the phase plane of (5.9) on which $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is of one sign .

Proof

We make the usual assumptions about the smoothness of the vector field χ necessary to justify the application of the divergence theorem . First suppose there is a closed phase path Γ in the region Ω where $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is of one sign .
Hence by Stokes' theorem

$$\iint_{\mathcal{S}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy = \int_{\Gamma} \chi.\mathbf{n}\, da \tag{5.10}$$

where $\mathcal{S}$ is the interior of Γ . Since on Γ , the vector field χ must be perpendicular to $\mathbf{n}$, the integral on right is zero . On the contrary the integrand on left is of one sign($i.e$ either positive or negative). So its integral cannot be zero . Therefore Γ cannot be a closed path .

5.4.2 Example

Consider the dynamical system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{5.11}$$

can have no periodic solution whose phase path lies in a region where f is of one sign .
The equivalent system is thus

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - g(x) \end{aligned} \tag{5.12}$$

Hence $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = -f(x)$
which is of one sign whenever f is of one sign .

[1] Any simple closed curve consisting of several singular points and the trajectories which approach the singular points as $t \to \pm\infty$ is called the singular closed trajectory of the given system.

5.4.3 Example

Examine whether the following system can have no periodic solutions

$$\begin{aligned} \dot{x} &= x(y^2+1)+y \\ \dot{y} &= (x^2+1)(y^2+1) \end{aligned} \tag{5.13}$$

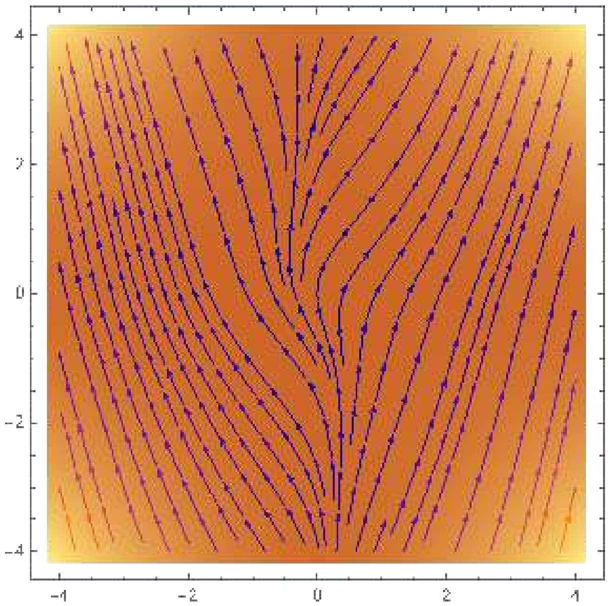

Figure 5.2: Phase portrait of the system 5.13

5.4.4 Another example

By plotting the phase portrait show that the dynamical system represented by the equation

$$\ddot{x} + 0.15\dot{x} + \sin x = 0 \tag{5.14}$$

for a *damped pendulum* does not have any closed trajectory . First write down the equivalent planar system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -0.15y - \sin x \end{aligned} \tag{5.15}$$

Now using the Bendixson negative criterion

$$\begin{aligned} &\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} \\ &= 0 - 0.15 \\ &< 0 \end{aligned}$$

is of only one sign at any cost . Hence , there cannot exist any closed trajectory of the given damped pendulum(see figure-5.3)

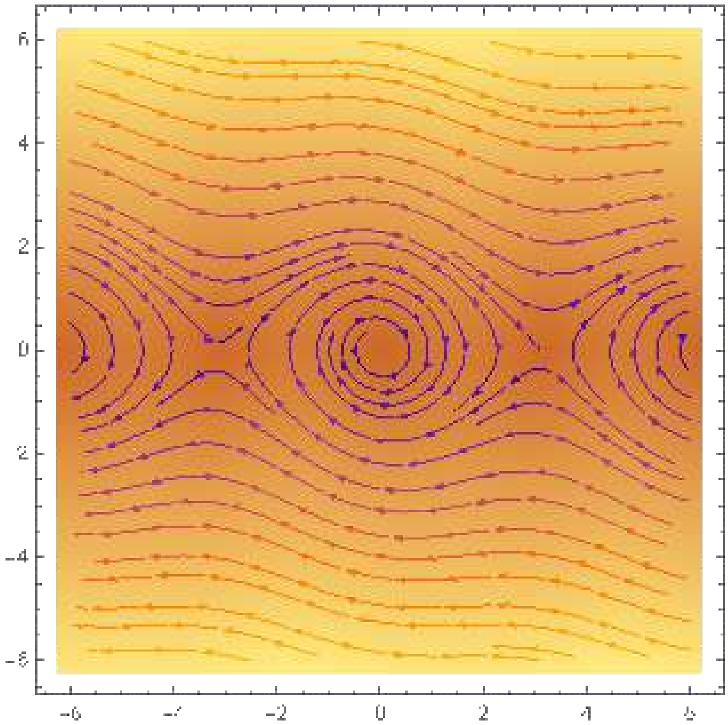

Figure 5.3: Phase portrait of the motion of a damped pendulum

5.4.5 Example

Discuss the limit cycles of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -4x - 5y \end{aligned} \tag{5.16}$$

We first project the phase paths of the given system onto the diametrical plane . This represents a *heavily damped or overdamped oscillator*[2] . We call the phase plane as P . As we can see that the origin is the only equilibrium point . Now compute

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} \\ = 0 - 5 \\ < 0$$

Using Bendixson negative criterion , there does not exist any closed path . So , no question of limit cycles arises .

[2]A system that returns to its equilibrium without any oscillation slowly also reduces the amplitude

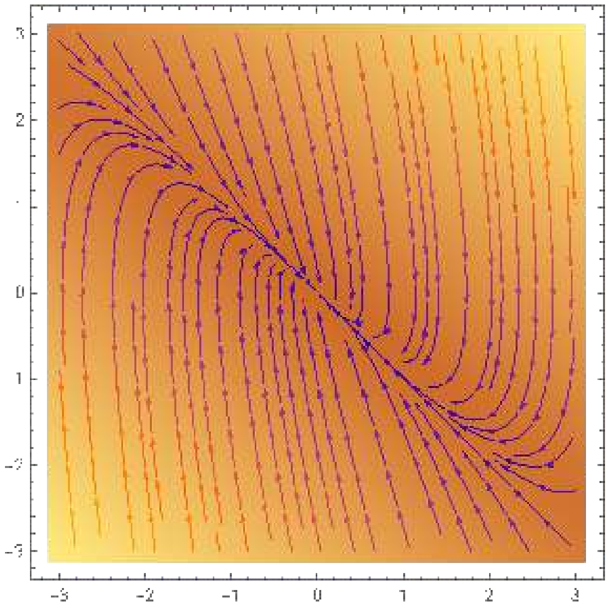

Figure 5.4: Phase portrait of the given system

5.5 Bendixson-Dulac's criterion

Let the C^∞-planar system

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned} \tag{5.17}$$

defined in some *simply connected* open neighborhood U of $\mathbb{R}^2$ and smooth vector field $\chi = (P,Q)$.
Define a C^∞ map
$\phi : U \to \mathbb{R}$ so that the divergence

$$\nabla \cdot (\phi\chi)|_U = \frac{\partial}{\partial x}((\phi(x,y)P(x,y)) + \frac{\partial}{\partial y}((\phi(x,y)Q(x,y)) \geq 0 \text{ (or } \leq 0) \tag{5.18}$$

and vanishes only on a set with Lebesgue measure zero . Then there does not exist any closed orbit in U .

5.5.1 Theorem

Consider the system

$$\begin{aligned} \dot{x} &= -y + \delta x + mxy - y^2 \\ \dot{y} &= x(1 + ax) \end{aligned} \tag{5.19}$$

System (5.19) cannot have a closed trajectory or the singular close trajectory passing a saddle point in either of the following cases :

1. $m\delta \leq 0$, $|m| + |\delta| \neq 0$

2. $\delta(m-\delta) \leq 0$, $|m| + |\delta| \neq 0$

Proof

When the first group of conditions hold , we see that any closed trajectory or any singular closed trajectory passing a saddle point of (19) cannot intersect the line $1 - mx = 0$. Now take the Dulac function

$$\phi = \frac{1}{1-mx} \tag{5.20}$$

, then we have

$$\frac{\partial}{\partial x}(\phi P) + \frac{\partial}{\partial y}(\phi Q) = \frac{\delta - my^2}{(1-mx)^2} \tag{5.21}$$

The right side of this relation always keeps a constant sign on both sides of the line $1 - mx = 0$, hence by Bendixson-Dulac criterion , first part is proved .
On the other hand suppose the second group of conditions also hold . Now translating the x-axis to the line $y = -\frac{\delta}{m}$ and keeping the y-axis unchanged. Then the equation (5.19) becomes

$$\begin{aligned} \dot{x} &= \tfrac{\delta}{m}(1-\tfrac{\delta}{m}) + (1+2\tfrac{\delta}{m})y + mxy - y^2 \\ \dot{y} &= x(1+ax) \end{aligned} \tag{5.22}$$

The system of equations whose vector field is symmetric to that of (22)

$$\begin{aligned} \dot{x} &= \tfrac{\delta}{m}(1-\tfrac{\delta}{m}) - (1+2\tfrac{\delta}{m})y - mxy - y^2 \\ \dot{y} &= -x(1+ax) \end{aligned} \tag{5.23}$$

The locus of points of contact of the trajectories of these two systems is easily seen to be

$$\begin{aligned} x &= 0 \\ 1+ax &= 0 \\ \tfrac{\delta}{m}(1-\tfrac{\delta}{m}) - y^2 &= 0 \end{aligned} \tag{5.24}$$

Now observe that the divergence of the last equation is zero on the excess and so any closed or singular closed trajectory Γ must intersect the x-axis. Also if Γ appears in the vicinity of $(0,0)$, then it cannot meet $1+ax=0$. Also for $\delta(m-\delta) < 0$ the last equation of the last system does not have a real locus . And for $\delta(m-\delta) = 0$ its locus is the x-axis .
we cannot see that the curve symmetric to Γ_x^+ (the portion of Γ above the x-axis) and Γ_x^- (the portion of Γ below the x-axis) cannot have a common point except on the x-axis , that is the curve symmetric to Γ_x^+ , lies entirely above or below Γ_x^- .
So , any closed or singular closed trajectory Γ of the last system we have

$$\iint_{\text{int }\Gamma} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy = \iint_{\text{int }\Gamma} -mydxdy \neq 0$$

This leads to a contradiction , so Γ does not exist.

5.5.2 Example

Show that the system

$$\begin{aligned} \dot{x} &= x(2-x-y) \\ \dot{y} &= y(4x-x^2-3) \end{aligned} \tag{5.25}$$

has no closed orbit for $x, y > 0$.
Define the smooth map $\phi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$\phi(x,y) = \frac{1}{xy} \tag{5.26}$$

Hence

$$\begin{aligned} \nabla \cdot (\phi\chi)|_{(0,\infty)} &= \frac{\partial}{\partial x}(\phi\chi) + \frac{\partial}{\partial x}(\phi\chi) \\ &= \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right) \\ &= -\frac{1}{y} \\ &< 0 \end{aligned}$$

Now using Bendixson-Dulac criterion since the finite product of open rays $(0,\infty) \times (0,\infty)$ having the standard product topology is evidently simply connected , the given system does not have any closed trajectory(see figure 5.5)

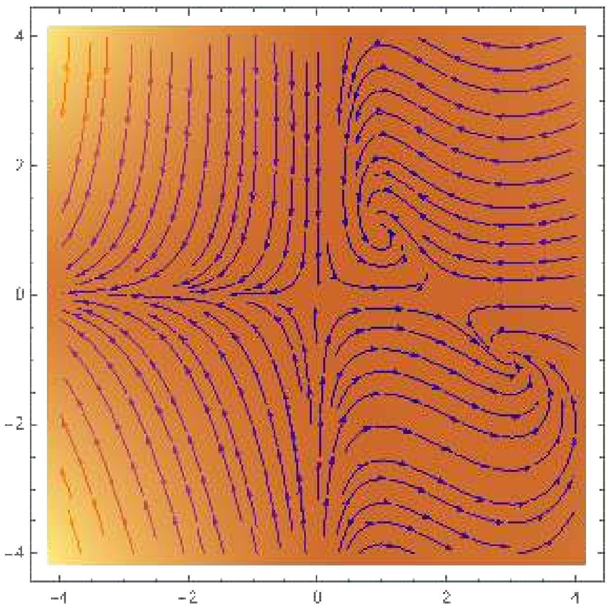

Figure 5.5: No closed trajectory for the system (5.25)

5.5.3 Example

Show that the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - y + x^2 + y^2 \end{aligned} \tag{5.27}$$

has no closed orbits .
Define a C^∞ map $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = e^{-2x} \tag{5.28}$$

So ,

$$\begin{aligned} \nabla \cdot (\phi\chi) &= -2e^{-2x}y + e^{-2x}(-1 + 2y) \\ &= -e^{-2x} \\ &< 0 \end{aligned}$$

Hence by Bendixson-Dulac criterion there does not exist any closed trajectory .

5.5.4 Definition

Consider the smooth system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \tag{5.29}$$

Consider the associated smooth vector field $\chi = P(x, y)\partial_x + Q(x, y)\partial_y$ and hence the $div\,\chi = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$
Now for an open set U in $\mathbb{R}^2$ a smooth function $V : U \to \mathbb{R}$ is said to be an *inverse integrating factor* of the system (5 29) if it is not locally null and satisfy the *PDE*

$$P(x, y)\frac{\partial V(x, y)}{\partial x} + Q(x, y)\frac{\partial V(x, y)}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V(x, y) \tag{5.30}$$

5.5.5 Proposition

Consider a C^∞ vector field of some system as $\chi = (P, Q)$ defined in an open set U of $\mathbb{R}^2$ admits an inverse integrating factor $V(z)$, where $z = f(x, y)$ if and only if

$$\frac{div\,\chi}{P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y}} = \alpha(z) \tag{5.31}$$

where $\alpha(z)$ is an exclusive function of z in such case the inverse integrating factor is of the form

$$V = \exp(\int_0^z \alpha(x)dx) \tag{5.32}$$

Proof

Assume that $V = V(z)$ where $z = f(x, y)$. We apply the chain rule in (5.30)

$$P\frac{dV}{dz}\frac{\partial f}{\partial x} + Q\frac{dV}{dz}\frac{\partial f}{\partial y} = V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \tag{5.33}$$

Then collecting $\frac{dV}{dz}$ and V together as

$$\frac{\frac{dV}{dz}}{V} = \frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y}} = \alpha(z) \tag{5.34}$$

Hence we finally solve for V as

$$V = e^{\int_0^z \frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y}}} = e^{\int_0^z \alpha(x)dx} \tag{5.35}$$

Hence the proof.

An exquisite example of this chapter

Consider the system

$$\begin{aligned} \dot{x} &= -x - 4xy - 5y^2 \\ \dot{y} &= 3x + 2y + y^2 \end{aligned} \tag{5.36}$$

Show that the system has no limit cycle in the domain $\mathbb{R}^2 \setminus \{(x, y) : x + y^2 = 0\}$ First we note that the domain $\Sigma = \mathbb{R}^2 \setminus \{(x, y) : x + y^2 = 0\}$ is not simply connected .

Assume that the system (5.36) has an inverse integrating factor which is a function of $x + y^2$. So , if we take $z = x + y^2$ and compute

$$(-x - 4xy - 5y^2)\frac{dV}{dz} + (3x + 2y + y^2)\frac{dV}{dz}(2y) = V(1 - 2y) \tag{5.37}$$

Now writing

$$\frac{\frac{dV}{dz}}{V} = \frac{1 - 2y}{(-1 + 2y)(x + y^2)} = -\frac{1}{x + y^2} = -\frac{1}{z} \tag{5.38}$$

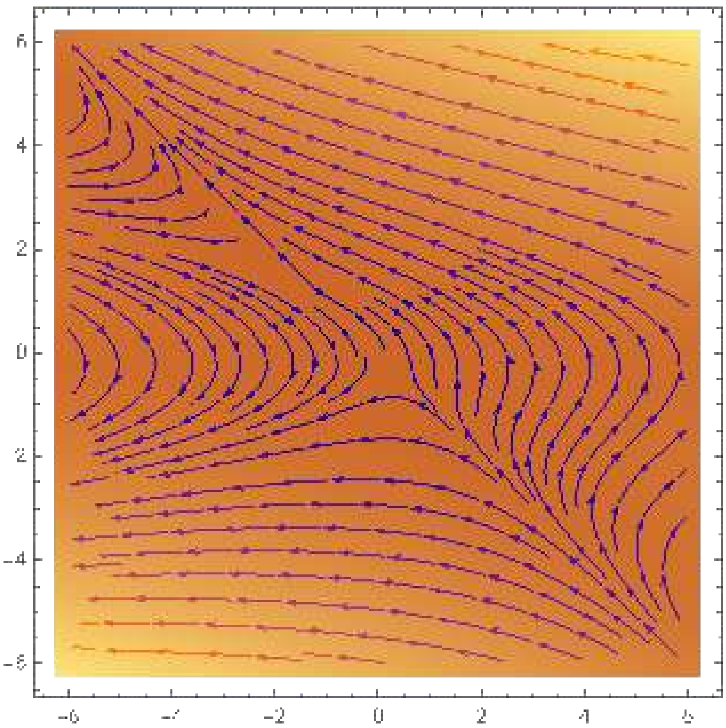

Figure 5.6: Phase portrait

Now integrating the equation

$$V = \frac{1}{z} = \frac{1}{x + y^2} \tag{5.39}$$

Hence on the domain $\Sigma = \mathbb{R}^2 \setminus \{(x, y) : x + y^2 = 0\}$ the system cannot have any limit cycle .

Chapter 6

Hilbert 16th Problem

6.1 Historical notes

Hilbert's 16th problem was first proposed by David Hilbert at the Paris conference of the International Congress of Mathematicians in 1900, as part of his list of 23 problems in mathematics. He upheld world's one of the most intriguing problem of the century by asking the position and the total number of limit cycles of a two dimensional planar system with polynomial vector fields .
This discussion on this chapter is unarguably one of the most crucial topics in the history of mathematics which is still mystery for many renowned and expert mathematicians so far .
Consider the system of differential equations

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned} \tag{6.1}$$

P and Q are purely polynomial functions (smooth) and the system must be in $\mathbb{R}^2$. Then Hilbert asked two important questions to the rest of the world -

1. First part

   In the first part of Hilbert 16th problem it asks about the position of the limit cycles or the algebraic curves in the planet system given by (6.1)

2. second part

   What could be the maximum number and positions of the limit cycles of the given system. We mainly focus on the second part of the Hilbert 16^{th} problem which is still bashing the world class mathematicians as the most heated topic ever .

We now try to simplify the above question.

Construct $\aleph_n = \{\#(P,Q) : deg\, P \text{ and } deg\, Q \leq n\}$, as the set of numbers of limit cycles of the system where $deg\, P = n = deg\, Q$ and $\#(P,Q)$ denotes the total number of limit cycles of the system (6.1) . Now define

$$H(n) = sup\, \aleph_n \tag{6.2}$$

$H(n)$ is known as *Hilbert Number*.Then our task will be to answer the question whether $sup\,\aleph_n$ exists or not . That is $H(n) < \infty$ or $H(n) = \infty$ for sufficiently large n .
The Hilbert number $H(n)$ is itself very intriguing in its nature . First take $n = 1$ *i.e.* , $H(1)$. Clearly it is a linear case .

6.2 Lemma

The Hilbert number $H(1) = 0$ *.*

Proof

Roughly speaking the above statement is trivial. Here $deg\ P \leq 1$ and $deg\ Q \leq 1$. First consider $deg\ (P, Q) = 0$ and thus it implies that

$$\dot{x} = k$$
$$\dot{y} = m$$

where $k, m \in \mathbb{R}$.
Then clearly the system cannot have any periodic orbit so no question of limit cycles arises .
Now consider the system

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$\forall a\ , b\ , c\ , d \in \mathbb{R}$
Here we have deg $P = 1 =$ deg Q . Now clearly the nature of the fixed points might be node , saddle , center or spiral . Of them only the center is a periodic orbit but not a limit cycle since it does have closed trajectories in its neighborhood .

6.2.1 Remark

1. System might have periodic orbits.
2. Due to the triviality of $H(1)$ mathematicians better begin with $H(n)$, $\forall n \geq 2$

6.2.2 A short background

The famous summer field wass the first person first obtained two limit cycles at once in 1929 . Later mathematician Bautin told that $H(2) \geq 3$ in 1939 .

The conjecture $H(2) = 3$

Afterwards Petrovski and Landis in 1955 first conjectured and later proved that $H(2) = 3$. But proof had few drawbacks discovered by another prominent mathematician Novikov and reproved their statement correctly in 1967 .

The conjecture $H(2) = 4$

$H(2) = 3$ was the conjecture till 1979 . But in the same year the Chinese mathematician Shi Songling constructed a system that clearly showed that $H(2) = 4$ which tumbled the world mathematicians in a moment . And this is still the unbreakable conjecture still today .
My next topic is deeply inspired by Dr.Shi Songling . It is just an experiment to see what happens if we consider the general equation of second degree as the polynomial having some specific values of four parameters.

6.3 Proposed approach

Consider the system

$$\dot{x} = a_1x^2 + b_1y^2 + 2h_1xy + 2f_1y + 2g_1x + c_1$$
$$\dot{y} = a_2x^2 + b_2y^2 + 2h_2xy + 2f_2y + 2g_2x + c_2$$

Treat a_i , b_i , c_i , h_i , g_i , f_i as fixed constants $\forall\, i \in \{1, 2\}$. Now choosing the values of the in terms of the parameters ϵ , μ , ν and ζ the system can be expressed in particular form as :

$$\dot{x} = 2x^2 + 3y^2 + 2\epsilon xy + 6y + 8(\nu + 9)x$$
$$\dot{y} = 3x^2 + 2y^2 + (\mu + 4\epsilon)xy - 8\zeta y + 6x$$

Choose the values of the parameters as

$$\epsilon = -10^{-14}$$
$$\mu = -10^{-300}$$
$$\nu = -0.008$$
$$\zeta = -10^{-500}.$$

Now putting the values of the parameters in the last system and plot using mathematica is to verify whether there is any limit cycle or not . The result in the figure 6.1 shows that there does not exist in a limit cycle .

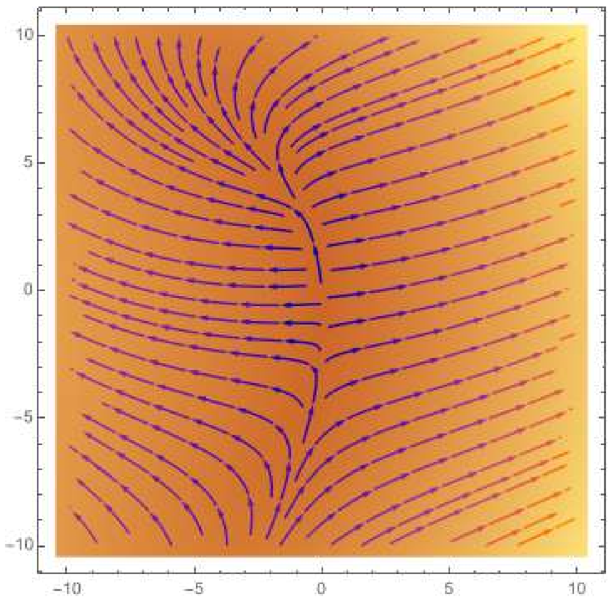

Figure 6.1: Phase portrait having no limit cycles

6.4 Poincaré compaction

Consider the system (6.1) again . By making a simple transformation it is possible to map the phase plane onto a unit-2-sphere . Interestingly, with the help of this map the phase plane can be mapped on both the upper and lower hemisphere .
Define a map $\pi_p^\beta : \mathbb{R}^2 \to \mathbb{E}_n^+$ by

$$\pi_p^\beta(x, y) = (X, Y, Z) \tag{6.3}$$

where (X, Y, Z) is a point on the upper hemisphere $\mathbb{E}_n^+$ (note that this could be done for lower hemisphere too) .
Now the given system can be transformed into the polar coordinates as

$$\begin{aligned} \dot{r} &= r^n f_{n+1}(\theta) + r^{n-1} f_{n-1}(\theta) + \cdots + f_1(\theta) \\ \dot{\theta} &= r^{n-1} g_{n+1}(\theta) + r^{n-2} g_{n-1}(\theta) + \cdots + r^{-1} g_1(\theta) \end{aligned} \tag{6.4}$$

where f_m and g_m are polynomials of degree m in $\cos\theta$ and $\sin\theta$.
Now put $\rho = \frac{1}{r}$. This gives $\dot{\rho} = -\frac{\dot{r}}{r^2}$. Hence

$$\begin{aligned} \dot{\rho} &= r^{n-1}[-\rho f_{n+1}(\theta) + O(\rho^2)] \\ \dot{\theta} &= r^{n-1}[g_{n+1}(\theta) + O(\rho)] \end{aligned} \tag{6.5}$$

The critical points at infinity can be found by solving the equations in (5)

$$\begin{aligned} \dot{\rho} &= 0 \\ \dot{\theta} &= 0 \end{aligned} \tag{6.6}$$

which is equivalent to solve

$$g_{n+1}(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta) \tag{6.7}$$

where P_n and Q_n are homogeneous polynomials of degree n .

6.4.1 Example

Show that the system

$$\begin{aligned} \dot{x} &= -\frac{x}{2} - y - x^2 + xy + y^2 \\ \dot{y} &= x(1 + x - 3y) \end{aligned} \tag{6.8}$$

has at least two limit cycles .

For the sake of simplicity Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y})$ denote the given vector field . The given system has two critical points at $(0, 0)$ and $(0, 1)$. The Jacobian is given by

$$J = \begin{pmatrix} -\frac{1}{2} - 2x + y & -1 + x + 2y \\ 1 + 2x - 3y & -3x \end{pmatrix}$$

Now $Jf(0,0) = \begin{pmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{pmatrix}$ and $Jf(1,0) = \begin{pmatrix} \frac{1}{2} & 1 \\ -2 & 0 \end{pmatrix}$
The equilibrium points at infinity must satisfy the equation $g_3(\theta) = 0$ that is

$$g_3(\theta) = \cos\theta Q_2(\cos\theta, \sin\theta) - \sin\theta P_2(\cos\theta, \sin\theta)$$

Hence from the given system it can written as

$$g_3(\theta) = \cos^3\theta - 2\cos^2\theta\sin\theta - \cos\theta\sin^2\theta - \sin^3\theta$$

The plot of $g_3(\theta)$ given below

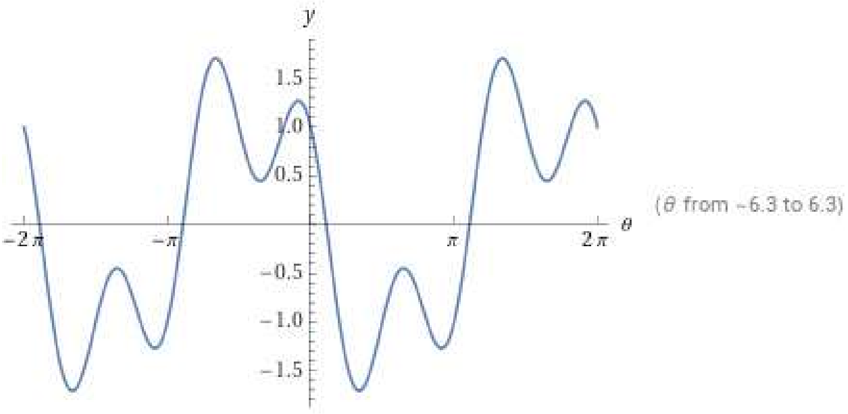

Figure 6.2: The plot of $g_3(\theta)$

It can be shown that there are two roots of the equation $g_3(\theta) = 0$. So there exist critical points at infinity. And the flow near those critical points at infinity is qualitatively equivalent to the flow of the system . The given phase portrait shows that there are two limit cycles(see figure 6.3).

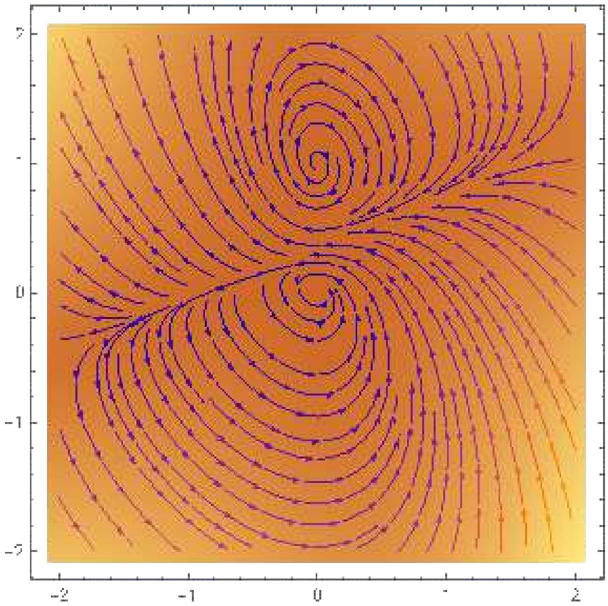

Figure 6.3: Phase portrait for the given system

6.4.2 Example

Consider the Liènard system

$$\ddot{x} + (-0.4 + 7.5x^2 - 5kx^4)\dot{x} + x = 0 \tag{6.9}$$

which does not have any limit cycle for any real values of k . We here chose only $k = 3.65$. Hence the phase portrait is shown in the figure 6.4 below

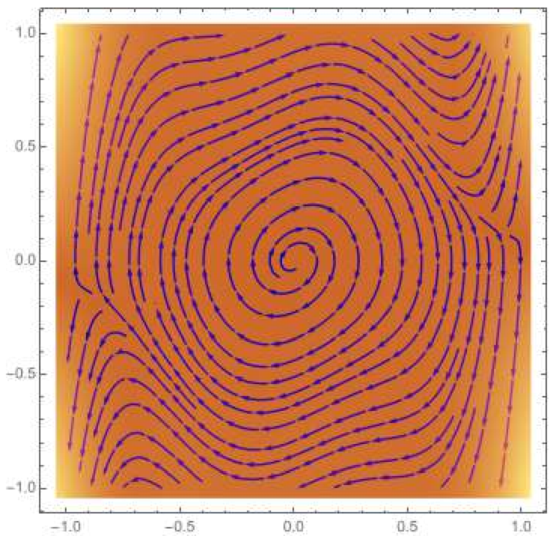

Figure 6.4: Phase portrait for $k = 3.65$

It can be seen that from the vector field plot that it is a center focus and after a certain time the trajectories are repelling outside .

Chapter 7

Conclusion

In this project, brief theory of limit cycle has been studied. Entire study has been divided into six chapters. In first chapter, concept of vector field and smooth function have been described. In this part, the inter relation between vector field and flow of dynamical system has been discussed.
In second chapter, concept of limit sets, attractors have been described on the basis of topological viewpoint. More interesting part, that is the strange attractor also have been discussed in the last part of this chapter.
Third chapter has been devoted to discuss the stability of limit cycles. In this section different examples have been given to analyze the stability of limit cycles.
Fourth chapter has been engaged to study the existence of limit cycles Here, different real life problems have been studied to explore the applications of limit cycles. An important theorem associated with stability of linearized model of nonlinear system has been presented.
Non existence of limit cycles have been studied in the fifth chapter. Here, Bendixson negative criterion and Bendixson-Dulac's criterion have been discuseed with the help of few examples .
In the final chapter, Hilbert's 16th Problem has been described. Still now , it is a most challenging problem. More research have been performed but there are many scope for future research in this area.
In entire project, limit cycles and the theory related existence and non existence have been presented. It is an interesting area of the dynamical system. However, it hoped that the project paper will be more helpful for the readers who are interested in area of dynamical system , specially on the theory of limit cycle.

Bibliography

[1] Theory Of Limit Cycles , Yan-Qian Ye , Translations Of Mathematical Monographs , Volume 66, American Mathematical Society , Translated by Chi Y.Lo , Michigan State University, copyright@1986.

[2] Differential Equations with Applications and Historical Notes, George F. Simmons , Third Edition , CRC Press , A Chapman & Hall Book,@2017.

[3] Dynamics In The Plane and The Poicare-Bendixson Theorem , Noah Geller , University of Chicago .

[4] The XVI-th Hilbert Problem About Limit Cycles, Henryk Zoladek , Institute of Mathematics , University of Warsaw , zoladek@mimuw.edu.pl , Panoramas of Mathematics , Banach Center Publications , Volume 34 , Institute of Mathematics , Polish Academy of Sciences , Waeszawa 1995.

[5] An introduction to mathematical optimal control theory spring , 2024 version by Lawrence C Evans , Department of Mathematics, University of California , Berkeley .

[6] Generic properties of singular trajectories , B. Bonnard, Universite de Bourgogne, Departement de Mathematiques, Laboratoire de Topologie, BP 138, 21004 Dijon Cedex, France. E-mail: bbonnard@satie.u-bourgogne.fr and I. Kupka University of Toronto, Department of Mathematics, M5S1Al Toronto, Canada.

[7] Picard's Existence and Uniqueness Theorem , Denise Gutermuth.

[8] Strange attractors , David Ruelle , Institut Des Hautes Etudes Scientifiques, F-91440 Bures-Yvette , France.

[9] Non-linear Oscillations , Dynamical Systems and Bifurcations Of Vector Fields, John Guckenheimer and Philip Holmes, Applied Mathematical Sciences 42,1985.

[10] Elementary Topics In Differential Geometry , John A . Thorpe , Springer-Verlag , New York Heidelberg Berlin,1979.

[11] Ordinary and Partial Differential Equations , An Introduction To Dynamical Systems, John W.Cain and Angela M.Reynolds,2010.

[12] Differential Equations and Dynamical Systems , Lawrence Perko , Third Edition , Text in Applied Mathematics 7 , Springer,2001.

[13] Inverse Integrating Factor for Classes of Planar Differential Equations , Khalil I.T. Al-Dosary , College of Sciences , University of Sharjah , P.O Box 27272 , Sharjah , United Arab Emirates , dosary@sharjah.ac.ae , Int.Journal of Math. Analysis ,Vol.4 , 2010 , no.29 ,1433-1446.

[14] Cauchy-Kovalevskaya Theorem, Bruce K.Driver , University of California San Diego , PDE lecture notes,Math 247A-B.

[15] Math 580 Lecture Notes 2 : THE Cauchy-Kovalevskaya Theorem , Tsogtgerel Gantumur , 2011 , McGill University.

[16] Topological conjugacy, december 8 , 2005 , Michigan State University.

[17] Hartman-Großman theorem , lecture note 3 , University of Nebraska-Lincoln.